Time Parallel Time Integration
Chapter 3: Domain Decomposition and Waveform Relaxation

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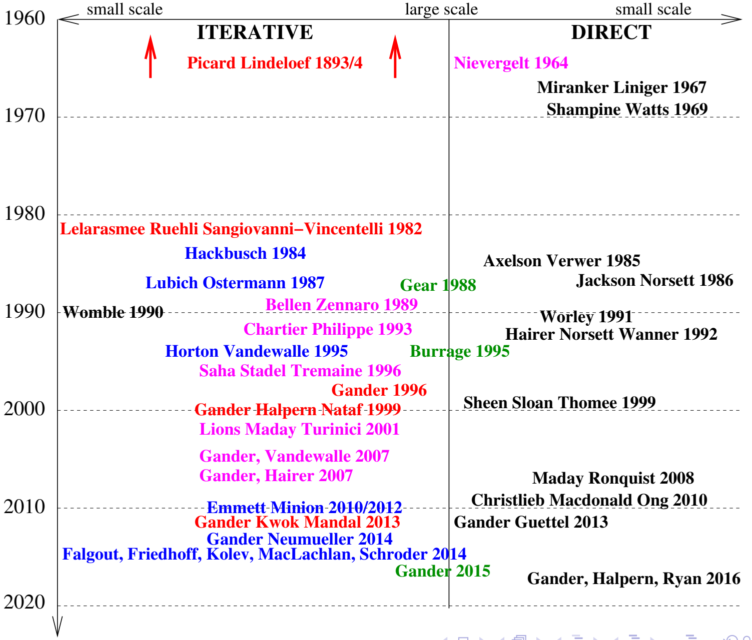
Domain Decomposition and Waveform Relaxation



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Quotes

Émile Picard (1883): Sur l'application des méthodes d'approximations successives à l'étude de certaines équations différentielles ordinaires,

> *"Les méthodes d'approximation dont nous faisons usage sont théoriquement susceptibles de s'appliquer à toute équation,* mais elles ne deviennent vraiment intéressantes *pour l'étude des propriétés des fonctions définies par les équations différentielles que* si l'on ne reste pas dans les généralités et si l'on envisage certaines classes d'équations"

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Quotes

Keith Miller (1965): Numerical Analogs to the Schwarz Alternating Procedure

> *"Schwarz's method presents some intriguing possibilities for numerical methods."*

Ekachai Lelarasmee and Albert E. Ruehli and Alberto L. Sangiovanni-Vincentelli (1982): The Waveform Relaxation Method for Time-Domain Analysis of Large Scale Integrated Circuits

> *"The spectacular growth in the scale of integrated circuits being designed in the VLSI era has generated the need for new methods of circuit simulation. "Standard" circuit simulators, such as SPICE and ASTAP, simply take too much CPU time and too much storage to analyze a VLSI circuit."*

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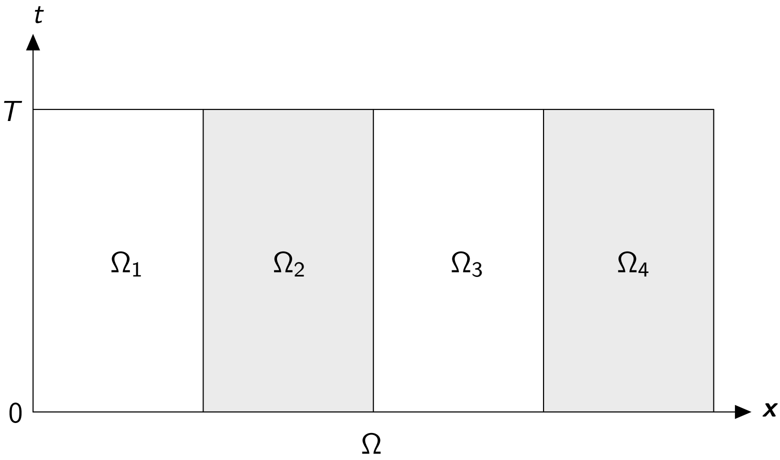
Typical decomposition for these methods



Decomposition of the space-time domain for time parallel methods based on domain decomposition

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Method of Successive Approximations

Ernest Lindelöf (1894): Sur l'application des méthodes d'approximations successives à l'étude des intégrales réelles des équations différentielles ordinaires

> *"La présente étude a pour but de donner une exposition succincte de la méthode d'approximations successives de M. Picard en tant qu'elle s'applique aux équations différentielles ordinaires"*

William Edmund Milne (1953): Numerical solution of differential equations

> *"Actually this method of continuing the computation is highly inefficient and is not recommended."*

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For Ordinary Differential equations

$$\begin{aligned} \partial_t \boldsymbol{u}(t) &= \boldsymbol{f}(t, \boldsymbol{u}(t)) \quad t \in (0, T], \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0, \end{aligned}$$

For existence, Picard writes the problem in integral form,

$$\boldsymbol{u}(t) = \boldsymbol{u}(0) + \int_0^t \partial_t \boldsymbol{u}(\tau) d\tau = \boldsymbol{u}_0 + \int_0^t \boldsymbol{f}(\tau, \boldsymbol{u}(\tau)) d\tau.$$

Picard iteration: compute for $k = 0, 1, 2, \ldots$

$$\boldsymbol{u}^{k+1}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{f}(\tau, \boldsymbol{u}^k(\tau)) d\tau.$$

A sequence of problems using only quadrature was much easier to handle at the time of Picard than an ODE.

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Convergence Analysis

Theorem (Lindelöf 1894: Superlinear Convergence)

If f is continuous, and uniformly Lipschitz with Lipschitz constant L in its second argument for all $t \in (0, T]$,

$$||\boldsymbol{f}(t, \boldsymbol{v}) - \boldsymbol{f}(t, \boldsymbol{w})|| \leq L||\boldsymbol{v} - \boldsymbol{w}|| \quad \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^d,$$

then the Picard iteration converges for any $\boldsymbol{u}^0(t)$ on bounded time intervals $t \in [0, T]$, and the iterates satisfy the superlinear error estimate

$$||\boldsymbol{u} - \boldsymbol{u}^k||_T \leq \frac{(LT)^k}{k!}||\boldsymbol{u} - \boldsymbol{u}^0||_T,$$

where $||\boldsymbol{u}||_T := \max_{0 \leq t \leq T} ||\boldsymbol{u}(t)||$ denotes the maximum norm in $[0, T]$.

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Proof.

Subtracting the Picard iteration from the integral form of the problem gives

$$\begin{aligned} \boldsymbol{u}(t) - \boldsymbol{u}^k(t) &= \boldsymbol{u}_0 + \int_0^t \boldsymbol{f}(\tau, \boldsymbol{u}(\tau))d\tau - \boldsymbol{u}_0 - \int_0^t \boldsymbol{f}(\tau, \boldsymbol{u}^{k-1}(\tau))d\tau \\ &= \int_0^t \boldsymbol{f}(\tau, \boldsymbol{u}(\tau)) - \boldsymbol{f}(\tau, \boldsymbol{u}^{k-1}(\tau))d\tau. \end{aligned}$$

We can now take the norm on both sides, and use the Lipschitz condition on $\boldsymbol{f}$,

$$\begin{aligned} ||\boldsymbol{u}(t) - \boldsymbol{u}^k(t)|| &\leq \int_0^t ||\boldsymbol{f}(\tau, \boldsymbol{u}(\tau)) - \boldsymbol{f}(\tau, \boldsymbol{u}^{k-1}(\tau))||d\tau \\ &\leq L \int_0^t ||\boldsymbol{u}(\tau) - \boldsymbol{u}^{k-1}(\tau)||d\tau. \end{aligned}$$

Since this inequality also holds for $k-1$, $k-2$ and so on, we can introduce it on the right, and obtain

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Proof continued

$$\|\boldsymbol{u}(t)-\boldsymbol{u}^k(t)\| \leq L\int_0^t L\int_0^\tau \|\boldsymbol{u}(\tau_2)-\boldsymbol{u}^{k-1}(\tau_2)\|d\tau_2 d\tau$$
$$\leq L^k\int_0^t\int_0^\tau\cdots\int_0^{\tau_{k-1}}\|\boldsymbol{u}(\tau_k)-\boldsymbol{u}^0(\tau_k)\|d\tau_k\dots d\tau_2 d\tau.$$

We can now take the maximum of the initial error $\|\boldsymbol{u}(\tau_k)-\boldsymbol{u}^0(\tau_k)\|$ in time out of the integral, and then start integrating one integral after the other,

$$\|\boldsymbol{u}(t)-\boldsymbol{u}^k(t)\| \leq L^k\int_0^t\int_0^\tau\cdots\int_0^{\tau_{k-1}} d\tau_k\dots d\tau_2 d\tau\|\boldsymbol{u}-\boldsymbol{u}^0\|_t$$
$$= L^k\int_0^t\int_0^\tau\cdots\int_0^{\tau_{k-2}} \tau_{k-1}d\tau_{k-1}\dots d\tau_2 d\tau\|\boldsymbol{u}-\boldsymbol{u}^0\|_t$$
$$= L^k\int_0^t\int_0^\tau\cdots\int_0^{\tau_{k-3}} \frac{\tau_{k-2}^2}{2}d\tau_{k-2}\dots d\tau_2 d\tau\|\boldsymbol{u}-\boldsymbol{u}^0\|_t$$
$$= L^k\int_0^t\int_0^\tau\cdots\int_0^{\tau_{k-4}} \frac{\tau_{k-3}^3}{3!}d\tau_{k-3}\dots d\tau_2 d\tau\|\boldsymbol{u}-\boldsymbol{u}^0\|_t$$

Proof continued

$$
\begin{aligned}
&\vdots \\
&= L^k \int_0^t \frac{\tau^{k-1}}{(k-1)!} d\tau ||\boldsymbol{u} - \boldsymbol{u}^0||_t \\
&= \frac{L^k t^k}{k!} ||\boldsymbol{u} - \boldsymbol{u}^0||_t .
\end{aligned}
$$

So we have shown that

$$
||\boldsymbol{u}(t) - \boldsymbol{u}^k(t)|| \leq \frac{L^k t^k}{k!} ||\boldsymbol{u} - \boldsymbol{u}^0||_t .
$$

The expression on the right is monotonically increasing in t, so we can bound it by setting $t := T$, and then taking the maximum in t on the left gives the result.

Remarks:

- Same type of term as for the parareal convergence estimate. (see quote of Saha, Stadel and Tremaine)
- Method is however not very efficient (see Milne quote)

Classical Waveform Relaxation

Ekachai Lelarasmee and Albert E. Ruehli and Alberto L. Sangiovanni-Vincentelli (1982): The Waveform Relaxation Method for Time-Domain Analysis of Large Scale Integrated Circuits, 1982

> *"The Waveform Relaxation (WR) method is an iterative method for analyzing nonlinear dynamical systems in the time domain. The method, at each iteration,* decomposes the system into several dynamical subsystems, each of which is analyzed for the entire given time interval*."*

Waveform relaxation methods were invented in the research laboratory of IBM in Yorktown Heights in 1982 for VLSI design, motivated by the extremely rapid growth of integrated circuits.

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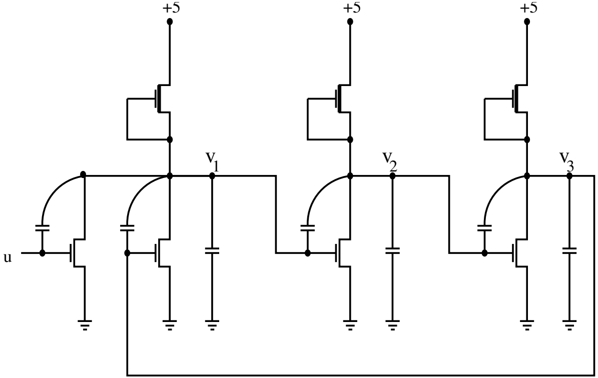
Historical Example: MOS ring oscillator



Using the laws of Ohm and Kirchhoff, the equations for such a circuit can be written in form of a system of ordinary differential equations,

$$
\begin{aligned}
\partial_t v_1(t) &= f_1(v_1(t), v_2(t), v_3(t)), \qquad v_1(0) = v_1^0,\\
\partial_t v_2(t) &= f_2(v_1(t), v_2(t), v_3(t)), \qquad v_2(0) = v_2^0,\\
\partial_t v_3(t) &= f_3(v_1(t), v_2(t), v_3(t)), \qquad v_3(0) = v_3^0.
\end{aligned}
$$

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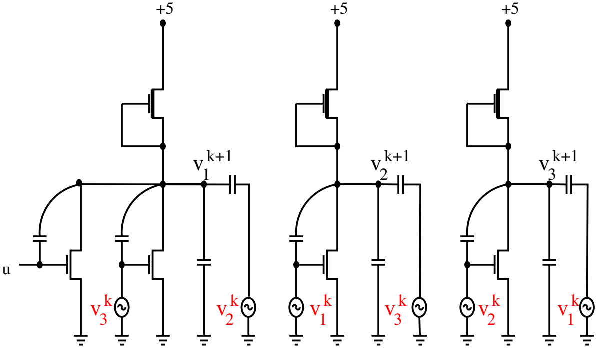
Decomposition of the MOS ring oscillator

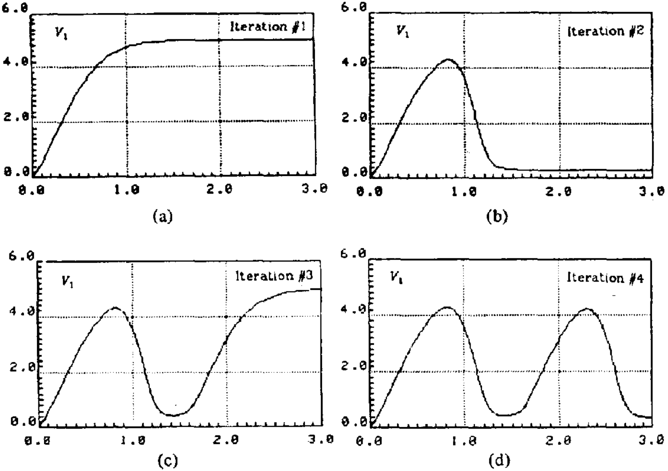


The *waveform relaxation* algorithm

$$\begin{aligned}
\partial_t v_1^k &= f_1(v_1^k, v_2^{k-1}, v_3^{k-1}), & v_1^k(0) &= v_1^0,\\
\partial_t v_2^k &= f_2(v_1^{k-1}, v_2^k, v_3^{k-1}), & v_2^k(0) &= v_2^0,\\
\partial_t v_3^k &= f_3(v_1^{k-1}, v_2^{k-1}, v_3^k). & v_3^k(0) &= v_3^0.
\end{aligned}$$

Start with some initial guess $v_1^0(t)$, $v_3^0(t)$, $v_3^0(t)$.

Historical Convergence Study from 1982



Ruehli et al: “Note that since the oscillator is highly non unidirectional due to the feedback from v_3 to the NOR gate, the convergence of the iterated solutions is achieved with the number of iterations being proportional to the number of oscillating cycles of interest”

Jacobi and Gauss-Seidel variants

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Instead of the Jacobi waveform relaxation algorithm

$$\begin{array}{rcll} \partial_t v_1^k & = & f_1(v_1^k, v_2^{k-1}, v_3^{k-1}), & v_1^k(0) = v_1^0, \\ \partial_t v_2^k & = & f_2(v_1^{k-1}, v_2^k, v_3^{k-1}), & v_2^k(0) = v_2^0, \\ \partial_t v_3^k & = & f_3(v_1^{k-1}, v_2^{k-1}, v_3^k). & v_3^k(0) = v_3^0. \end{array}$$

one could also use a Gauss-Seidel variant

$$\begin{array}{rcll} \partial_t v_1^k & = & f_1(v_1^k, v_2^{k-1}, v_3^{k-1}), & v_1^k(0) = v_1^0, \\ \partial_t v_2^k & = & f_2(v_1^k, v_2^k, v_3^{k-1}), & v_2^k(0) = v_2^0, \\ \partial_t v_3^k & = & f_3(v_1^k, v_2^k, v_3^k). & v_3^k(0) = v_3^0. \end{array}$$

How can one define in general partition functions for systems of ODEs?

Convergence Analysis

$$\begin{aligned} \partial_t \boldsymbol{u}(t) &= \boldsymbol{f}(t, \boldsymbol{u}(t)) \quad t \in (0, T], \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0. \end{aligned}$$

Introduce a partition function $\tilde{\boldsymbol{f}}(t, \boldsymbol{v}, \boldsymbol{w})$ such that

$$\tilde{\boldsymbol{f}}(t, \boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{f}(t, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{R}^d, \ t \in (0, T].$$

For an initial guess $\boldsymbol{u}^0(t)$, WR computes for $k = 0, 1, 2, \ldots$

$$\begin{aligned} \partial_t \boldsymbol{u}^{k+1}(t) &= \tilde{\boldsymbol{f}}(t, \boldsymbol{u}^{k+1}(t), \boldsymbol{u}^k(t)) \quad t \in (0, T], \\ \boldsymbol{u}^{k+1}(0) &= \boldsymbol{u}_0. \end{aligned}$$

- White, Odeh, Sangiovanni-Vincentelli, Ruehli (1985): linear convergence estimate.
- Nevanlinna (1989): superlinear estimate for linear matrix splittings $A = M - N$
- Bellen, Zennaro (1993): “By some standard analysis, we can easily get…”
- Burrage (1993): “it is easy to prove”.
- Vandewalle (1993): linear convergence using weighted norms.
- In't Hout (1995): quotes a convergence estimate citing Bellen et al, Burrage, Nevanlinna, and Lindelöf.

Gronwall lemma in integral form

Lemma (Gronwall Lemma (1919))

Let $u(t)$, $\alpha(t)$ and $\beta(t)$ be continuous functions on $[0,T]$. If $\beta(t) \geq 0$ and

$$u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s)ds \quad \forall t \in [0,T],$$

then

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \beta(\tau)d\tau}ds \quad \forall t \in [0,T].$$

Proof.

Exercise. □

We can now give an elementary general convergence proof for waveform relaxation methods.

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General convergence result

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Theorem (Superlinear Convergence)

If the partition function $\tilde{\boldsymbol{f}}(t, \boldsymbol{v}, \boldsymbol{w})$ is Lipschitz continuous in both arguments uniformly for all $t \in [0, T]$,

$$\begin{aligned} \|\tilde{\boldsymbol{f}}(t, \boldsymbol{v}_1, \boldsymbol{w}) - \tilde{\boldsymbol{f}}(t, \boldsymbol{v}_2, \boldsymbol{w})\| &\leq L_1 \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|, \\ \|\tilde{\boldsymbol{f}}(t, \boldsymbol{v}, \boldsymbol{w}_1) - \tilde{\boldsymbol{f}}(t, \boldsymbol{v}, \boldsymbol{w}_2)\| &\leq L_2 \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|, \end{aligned}$$

then the waveform relaxation algorithm satisfies the error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}^k\|_T \leq e^{L_1 T} \frac{(L_2 T)^k}{k!} \|\boldsymbol{u} - \boldsymbol{u}^0\|_T,$$

where $\|\boldsymbol{u}\|_T := \max_{0 \leq t \leq T} \|\boldsymbol{u}(t)\|$ *denotes again the maximum norm in* $[0, T]$.

Proof.

Subtract the integral form of the waveform relaxation iteration from the integral form of the problem,

$$\boldsymbol{u}(t) - \boldsymbol{u}^k(t) = \int_0^t \boldsymbol{f}(s,\boldsymbol{u}(s)) - \tilde{\boldsymbol{f}}(s,\boldsymbol{u}^k(s),\boldsymbol{u}^{k-1}(s))ds.$$

We now use that the partition function satisfies $\boldsymbol{f}(s,\boldsymbol{u}(s)) = \tilde{\boldsymbol{f}}(s,\boldsymbol{u}(s),\boldsymbol{u}(s))$, and adding and subtracting the term $\tilde{\boldsymbol{f}}(s,\boldsymbol{u}^k(s),\boldsymbol{u}(s))$, we get

$$\begin{aligned}\boldsymbol{u}(t) - \boldsymbol{u}^k(t) = & \int_0^t \tilde{\boldsymbol{f}}(s,\boldsymbol{u}(s),\boldsymbol{u}(s)) - \tilde{\boldsymbol{f}}(s,\boldsymbol{u}^k(s),\boldsymbol{u}(s))ds \\ & + \int_0^t \tilde{\boldsymbol{f}}(s,\boldsymbol{u}^k(s),\boldsymbol{u}(s)) - \tilde{\boldsymbol{f}}(s,\boldsymbol{u}^k(s),\boldsymbol{u}^{k-1}(s))ds.\end{aligned}$$

We can thus take the norm on both sides and use the Lipschitz conditions (similarity with parareal proof)

$$||\boldsymbol{u}(t)-\boldsymbol{u}^k(t)|| \le L_1\int_0^t ||\boldsymbol{u}(s)-\boldsymbol{u}^k(s)||ds + L_2\int_0^t ||\boldsymbol{u}(s)-\boldsymbol{u}^{k-1}(s)||ds.$$

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Proof continued

Setting $\beta(t) := L_1$ and $\alpha(t) := L_2 \int_0^t ||\boldsymbol{u}(s) - \boldsymbol{u}^{k-1}(s)||ds$, we can apply the Gronwall Lemma and obtain

$$||\boldsymbol{u}(t) - \boldsymbol{u}^k(t)|| \leq L_2 \int_0^t ||\boldsymbol{u}(s) - \boldsymbol{u}^{k-1}(s)||ds$$
$$+L_1 L_2 \int_0^t \int_0^s ||\boldsymbol{u}(\tau) - \boldsymbol{u}^{k-1}(\tau)||d\tau e^{L_1(t-s)} ds.$$

We now want to show by induction on k that the bound holds. For $k = 1$, we obtain by taking the maximum of the norms out of the integrals, estimating s by its upper bound t, and integration

$$||\boldsymbol{u}(t) - \boldsymbol{u}^1(t)|| \leq L_2 \int_0^t ||\boldsymbol{u}(s) - \boldsymbol{u}^0(s)||ds$$
$$+L_1 L_2 \int_0^t \int_0^s ||\boldsymbol{u}(\tau) - \boldsymbol{u}^0(\tau)||d\tau e^{L_1(t-s)} ds$$
$$\leq L_2 ||\boldsymbol{u} - \boldsymbol{u}^0||_t \left(t + L_1 \int_0^t s e^{L_1(t-s)} ds \right)$$

Proof continued

$$
\begin{aligned}
&\leq \quad L_2||\boldsymbol{u}-\boldsymbol{u}^0||_t\left(t+L_1\int_0^t se^{L_1(t-s)}ds\right)\\
&\leq \quad L_2||\boldsymbol{u}-\boldsymbol{u}^0||_t\left(t+L_1t\int_0^t e^{L_1(t-s)}ds\right)\\
&= \quad L_2||\boldsymbol{u}-\boldsymbol{u}^0||_t\left(t-t\left(1-e^{L_1t}\right)\right)=L_2te^{L_1t}||\boldsymbol{u}-\boldsymbol{u}^0||_t.
\end{aligned}
$$

We thus obtain for all $t \in [0, T]$ that

$$||\boldsymbol{u}(t)-\boldsymbol{u}^1(t)|| \leq L_2te^{L_1t}||\boldsymbol{u}(t)-\boldsymbol{u}^0(t)||_t \leq L_2Te^{L_1T}||\boldsymbol{u}-\boldsymbol{u}^0||_T,$$

and taking the maximum on the left concludes for $k=1$. So assume that the bound holds for $k-1$, and we show it for k: inserting this induction hypothesis into

$$
\begin{aligned}
||\boldsymbol{u}(t)-\boldsymbol{u}^k(t)|| \leq & L_2\int_0^t ||\boldsymbol{u}(s)-\boldsymbol{u}^{k-1}(s)||ds\\
&+L_1L_2\int_0^t\int_0^s ||\boldsymbol{u}(\tau)-\boldsymbol{u}^{k-1}(\tau)||d\tau e^{L_1(t-s)}ds,
\end{aligned}
$$

Proof continuted

we obtain by taking the maximum of the norms out of the integrals, switching the order of integration in the second term, and canceling two terms after integration

$$
\begin{aligned}
||\boldsymbol{u}(t)-\boldsymbol{u}^k(t)|| &\leq L_2\int_0^t e^{L_1 s}\frac{(L_2 s)^{k-1}}{(k-1)!}||\boldsymbol{u}-\boldsymbol{u}^0||_s ds\\
&\quad +L_1L_2\int_0^t\int_0^s e^{L_1\tau}\frac{(L_2\tau)^{k-1}}{(k-1)!}||\boldsymbol{u}-\boldsymbol{u}^0||_\tau d\tau e^{L_1(t-s)}ds\\
&\leq L_2||\boldsymbol{u}-\boldsymbol{u}^0||_t\left(\int_0^t e^{L_1 s}\frac{(L_2 s)^{k-1}}{(k-1)!}ds+L_1\int_0^t\int_0^s e^{L_1\tau}\frac{(L_2\tau)^{k-1}}{(k-1)!}d\tau e^{L_1(t-s)}ds\right)\\
&= L_2||\boldsymbol{u}-\boldsymbol{u}^0||_t\left(\int_0^t e^{L_1 s}\frac{(L_2 s)^{k-1}}{(k-1)!}ds+L_1\int_0^t\int_\tau^t e^{L_1(t-s)}ds e^{L_1\tau}\frac{(L_2\tau)^{k-1}}{(k-1)!}d\tau\right)\\
&= L_2||\boldsymbol{u}-\boldsymbol{u}^0||_t\left(\int_0^t e^{L_1 s}\frac{(L_2 s)^{k-1}}{(k-1)!}ds-\int_0^t\left(1-e^{L_1(t-\tau)}\right)e^{L_1\tau}\frac{(L_2\tau)^{k-1}}{(k-1)!}d\tau\right)\\
&= L_2||\boldsymbol{u}-\boldsymbol{u}^0||_t e^{L_1 t}\int_0^t\frac{(L_2\tau)^{k-1}}{(k-1)!}d\tau \;=\; e^{L_1 t}\frac{(L_2 t)^k}{k!}||\boldsymbol{u}-\boldsymbol{u}^0||_t,
\end{aligned}
$$

which concludes the proof by monotonicity in t of the right.

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Waveform Relaxation with Domain Decomposition

Juan Camilo Meza and William W. Symes (1987): Domain Decomposition Algorithms for linear Hyperbolic Equations

"Much of the current work in the application of domain decomposition techniques has been in the area of elliptic partial differential equations, with very little attention being given to hyperbolic equations [...] We take as our model problem the Dirichlet initial/boundary value problem for the one dimensional wave equation. We shall subdivide this problem into problems on smaller subdomains and synthesize the global solution out of the subdomain solutions, using the *finite propagation speed* and *superposition* properties of solutions."

G (1996): Overlapping Schwarz for linear and nonlinear parabolic problems

"Motivated by the work of Bjørhus (1995), we show how one can use *overlapping domain decomposition to obtain a waveform relaxation algorithm* for the semi-discrete heat equation which converges at a rate independent of the mesh parameter."

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Schwarz Waveform Relaxation, Heat Equation

$$
\begin{aligned}
\partial_t u(x,t) &= \partial_{xx} u(x,t) + f(x,t) && \text{in } \Omega \times (0,T],\ \Omega := (0,L),\\
u(x,0) &= u_0(x) && \text{in } \Omega,\\
u(0,t) &= g_0(t) && \text{in } (0,T],\\
u(L,t) &= g_L(t) && \text{in } (0,T].
\end{aligned}
$$

In analogy to the circuits, we partition the domain Ω into overlapping subdomains $\Omega_1 = (0,\beta)$ and $\Omega_1 = (\alpha, L)$, $\alpha < \beta$. The parallel Schwarz waveform relaxation relaxation algorithm then computes for $k = 0, 1, 2, \ldots$

$$
\begin{aligned}
\partial_t u_1^{k+1}(x,t) &= \partial_{xx} u_1^{k+1}(x,t) + f(x,t) && \text{in } \Omega_1 \times (0,T],\\
u_1^{k+1}(x,0) &= u_0(x) && \text{in } \Omega_1,\\
u_1^{k+1}(0,t) &= g_0(t) && \text{in } (0,T],\\
u_1^{k+1}(\beta,t) &= u_2^{k}(\beta,t) && \text{in } (0,T],
\end{aligned}
$$

$$
\begin{aligned}
\partial_t u_2^{k+1}(x,t) &= \partial_{xx} u_2^{k+1}(x,t) + f(x,t), && \text{in } \Omega_2 \times (0,T],\\
u_2^{k+1}(x,0) &= u_0(x) && \text{in } \Omega_2,\\
u_2^{k+1}(L,t) &= g_L(t) && \text{in } (0,T],\\
u_2^{k+1}(\alpha,t) &= u_1^{k}(\alpha,t) && \text{in } (0,T].
\end{aligned}
$$

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Initial guess, alternating variant

Need an initial guess for the solution at $x = \alpha$ and $x = \beta$, i.e. $u_1^0(\alpha, t)$ and $u_2^0(\beta, t)$ to start the iteration.

The alternating Schwarz waveform relaxation algorithm is very similar, one just has to replace the last interface update by

$$u_2^{k+1}(\alpha, t) = u_1^{k+1}(\alpha, t) \quad \text{in } (0, T],$$

but then the iteration can not be performed in parallel any more on the two subdomains.

The name Schwarz waveform relaxation comes from the fact that the decomposition is overlapping like in the classical overlapping Schwarz method for steady problems, and time dependent problems are solved in each iteration like in waveform relaxation.

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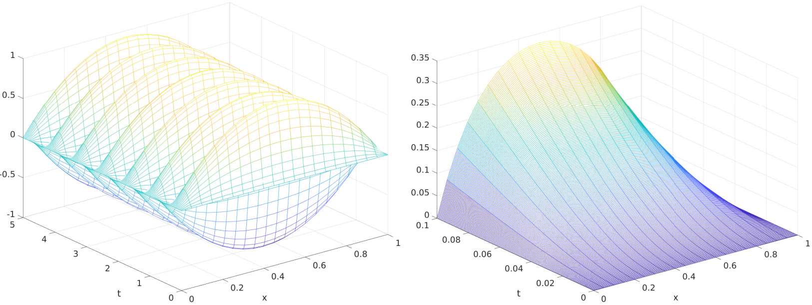
Numerical Experiment

Heat equation with $f(x,t) := x^4(1-x)^4 + 10\sin(8t)$



Solution over a long time interval $T = 5$ (left) and a short time interval $T = 0.1$ (right).

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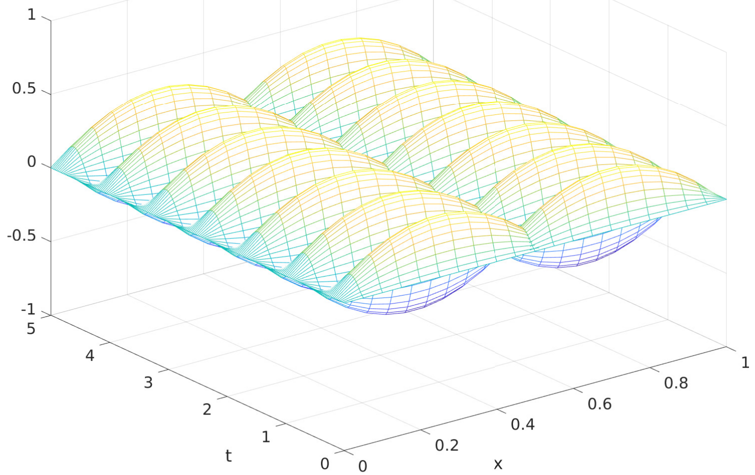
Schwarz WR Iteration 1 $T = 5$



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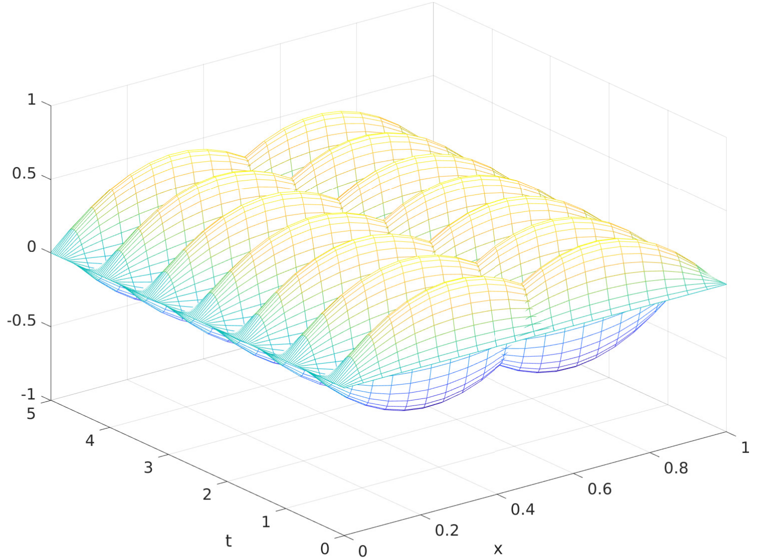
Schwarz WR Iteration 2 $T = 5$



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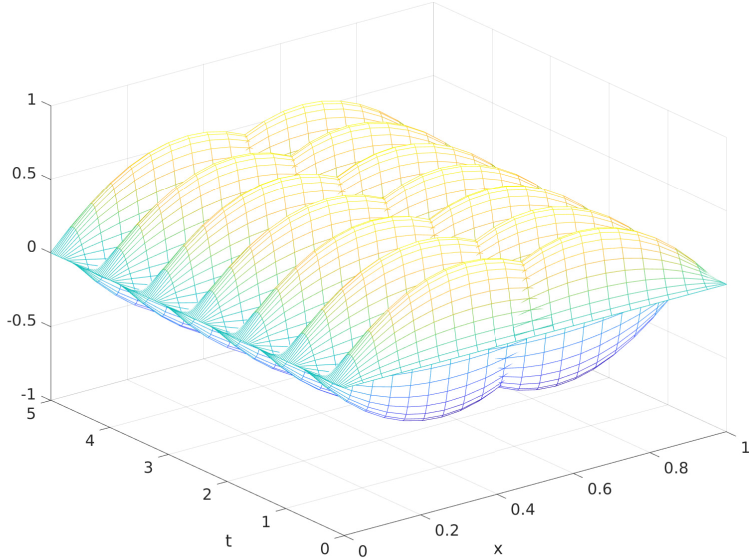
Schwarz WR Iteration 3 $T = 5$



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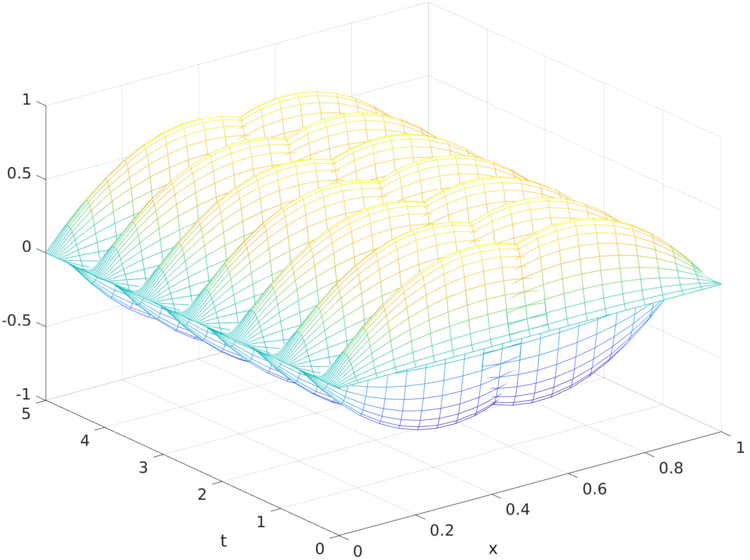
Schwarz WR Iteration 4 $T = 5$



PinT Summer School

Martin J. Gander

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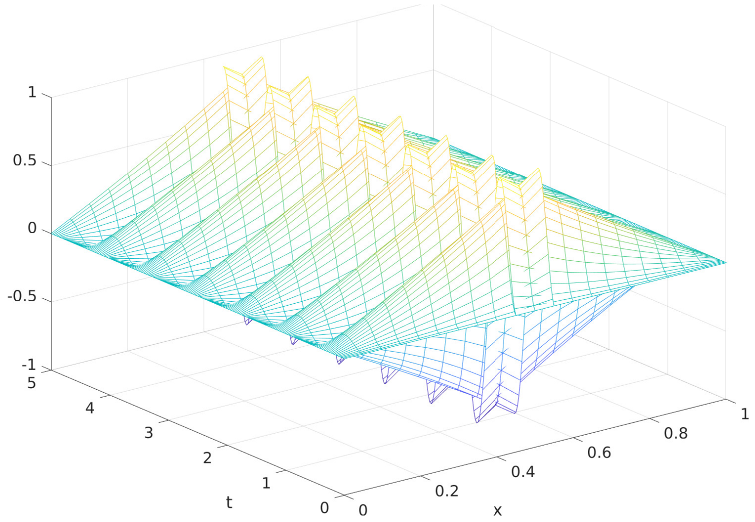
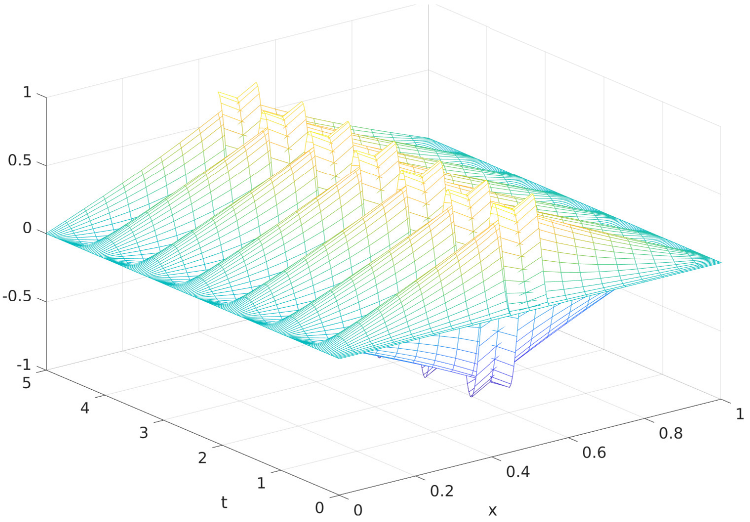
Schwarz WR Error Iteration 1 $T = 5$



Schwarz WR Error Iteration 2 $T = 5$



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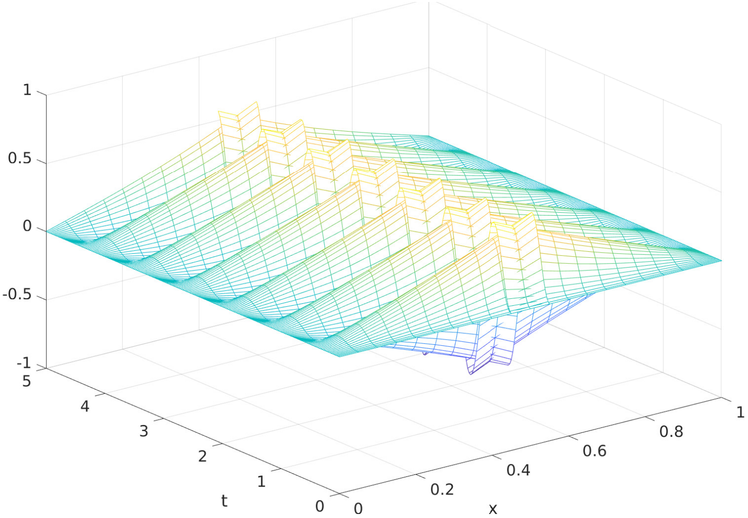

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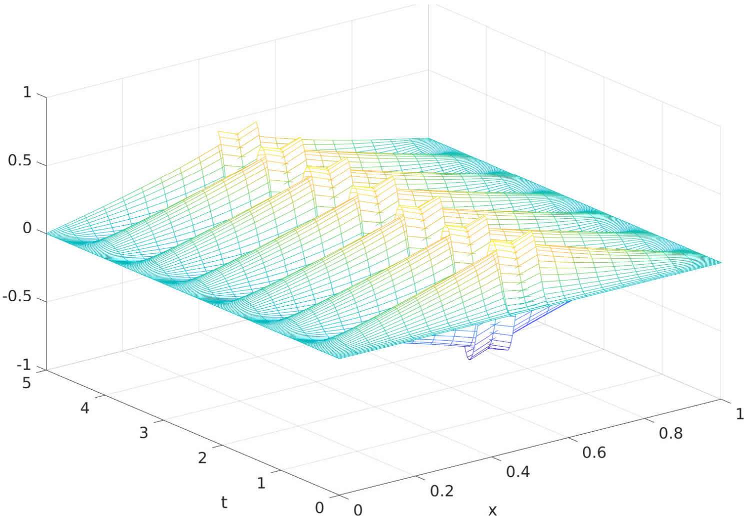
Schwarz WR Error Iteration 4 $T = 5$



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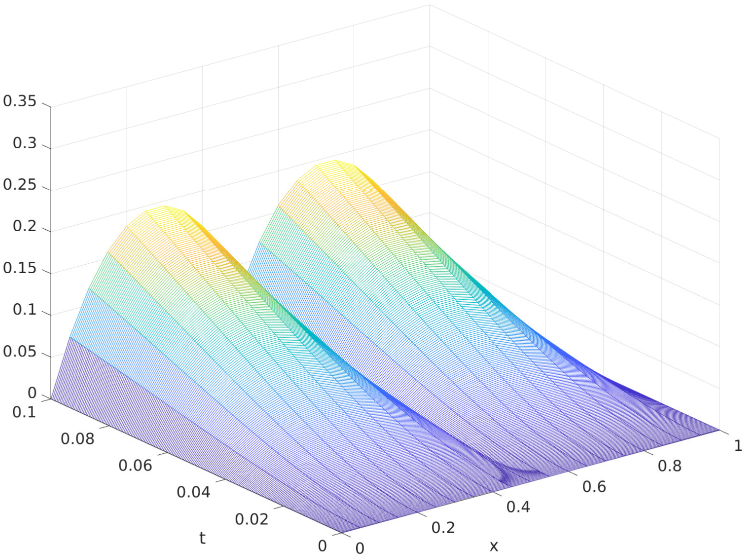
Schwarz WR Iteration 1 $T = 0.1$



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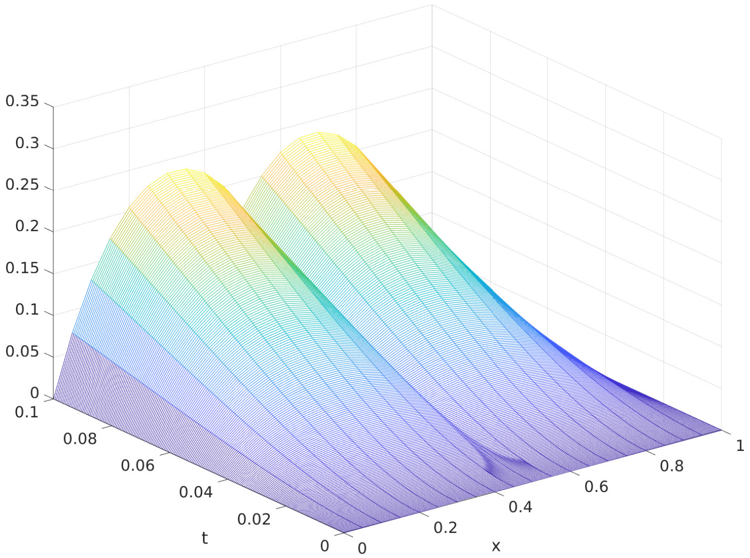
Schwarz WR Iteration 2 $T = 0.1$



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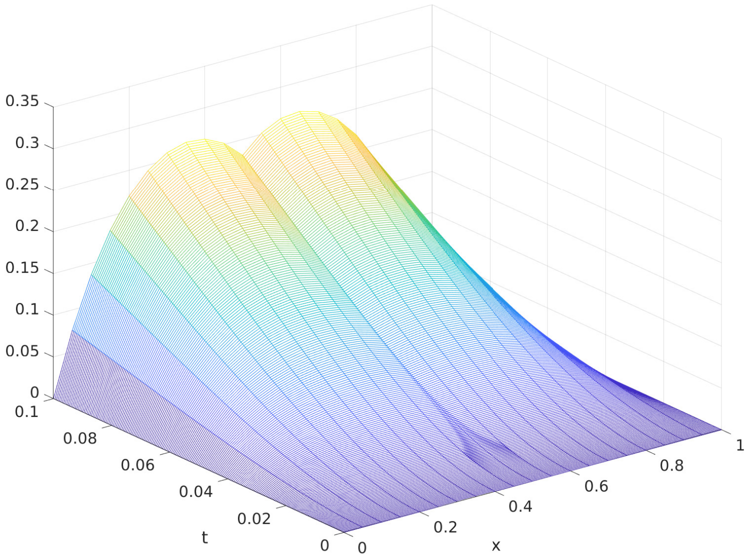

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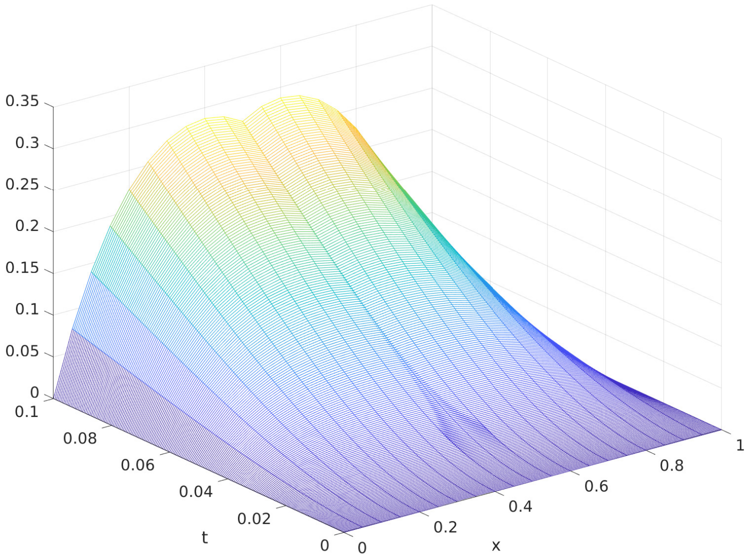
Schwarz WR Iteration 4 $T = 0.1$



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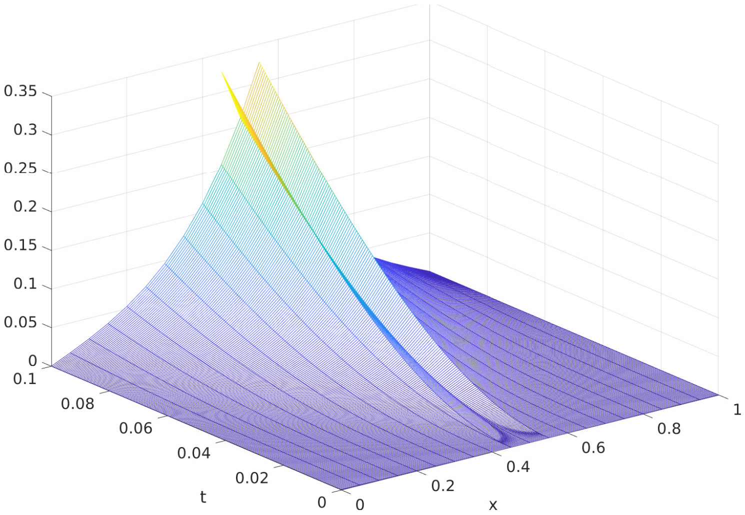
Schwarz WR Error Iteration 1 $T = 0.1$



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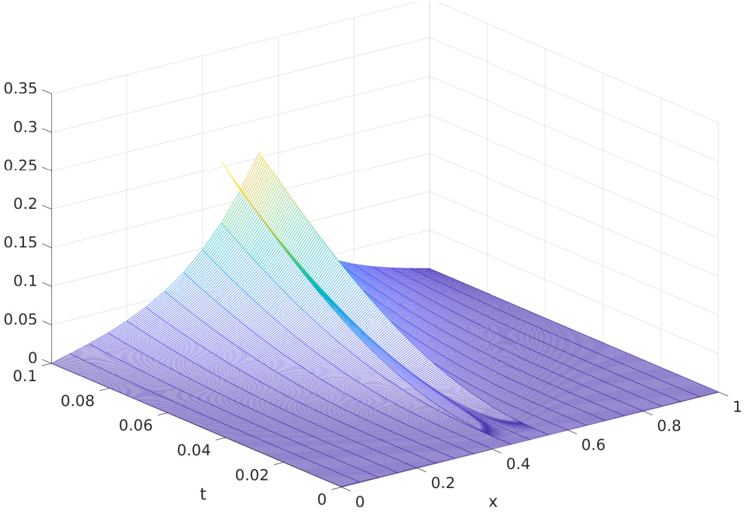
Schwarz WR Error Iteration 2 $T = 0.1$



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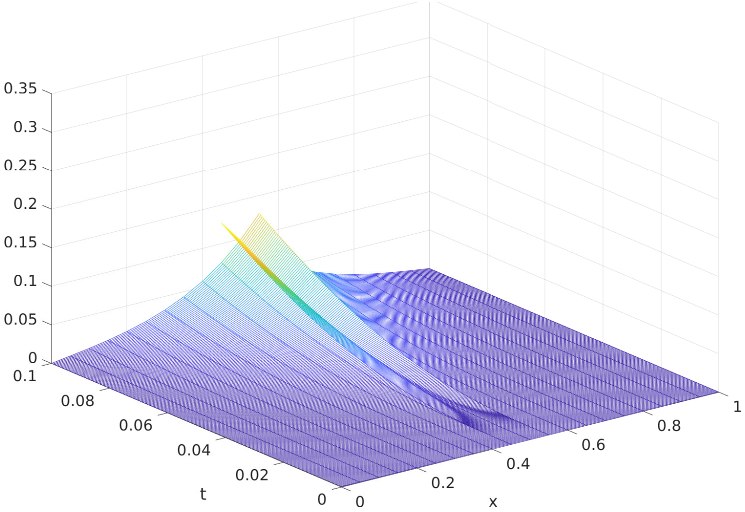

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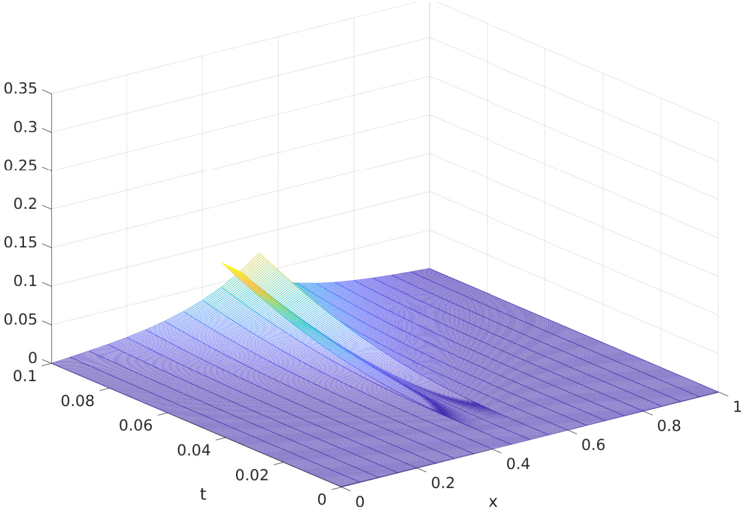
Schwarz WR Error Iteration 4 $T = 0.1$



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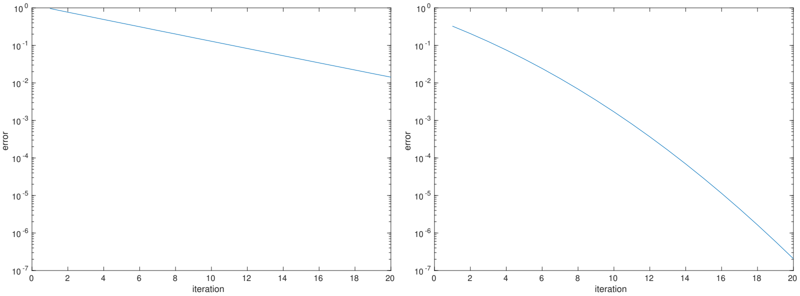
Error Decay as a Function of Iterations



Left: Schwarz waveform relaxation over the long time interval: linear convergence

Right: Schwarz waveform relaxation over the short time interval: superlinear convergence

Error Equations

The error $e_j^k(x,t) := u(x,t) - u_j^k(x,t)$ satisfies the homogeneous PDE

$$\begin{aligned}\partial_t e_j^k &= \partial_t(u(x,t) - u_j^k(x,t)) \\ &= \partial_{xx}u(x,t) + f(x,t) - \partial_{xx}u_j^k(x,t) - f(x,t) \\ &= \partial_{xx}e_j^k(x,t).\end{aligned}$$

The initial condition for the error is zero,

$$e_j^k(x,0) = u(x,0) - u_j^k(x,0) = u^0(x) - u^0(x) = 0,$$

and also on the original boundaries of the domain the error vanishes,

$$e_1^k(0,t) = u(0,t) - u_1^k(0,t) = g_0(t) - g_0(t) = 0,$$

and

$$e_2^k(L,t) = u(L,t) - u_2^k(L,t) = g_L(t) - g_L(t) = 0.$$

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Convergence Analysis

Iteration for the error:

$$
\begin{array}{rcll}
\partial_t e_1^{k+1}(x,t) &=& \partial_{xx} e_1^{k+1}(x,t) & \text{in } \Omega_1 \times (0,T],\\
e_1^{k+1}(x,0) &=& 0 & \text{in } \Omega_1,\\
e_1^{k+1}(0,t) &=& 0 & \text{in } (0,T],\\
e_1^{k+1}(\beta,t) &=& e_2^{k}(\beta,t) & \text{in } (0,T],\\
\\
\partial_t e_2^{k+1}(x,t) &=& \partial_{xx} e_2^{k+1}(x,t), & \text{in } \Omega_2 \times (0,T],\\
e_2^{k+1}(x,0) &=& 0 & \text{in } \Omega_2,\\
e_2^{k+1}(L,t) &=& 0 & \text{in } (0,T],\\
e_2^{k+1}(\alpha,t) &=& e_1^{k}(\alpha,t) & \text{in } (0,T].
\end{array}
$$

Need to study how the errors e_j^k converge to zero over long and short time intervals!

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Steady State Bounds for Long Time

Consider the steady state problems for fixed t

$$\begin{array}{rcll} \partial_{xx}\tilde{e}_1^{k+1}(x) & = & 0 & \text{in } \Omega_1, \\ \tilde{e}_1^{k+1}(0) & = & 0, & \\ \tilde{e}_1^{k+1}(\beta) & = & \|e_1^{k+1}(\beta,\cdot)\|_\infty, & \end{array}$$

$$\begin{array}{rcll} \partial_{xx}\tilde{e}_2^{k+1}(x) & = & 0, & \text{in } \Omega_2, \\ \tilde{e}_2^{k+1}(L) & = & 0, & \\ \tilde{e}_2^{k+1}(\alpha) & = & \|e_2^{k+1}(\alpha,\cdot)\|_\infty. & \end{array}$$

Lemma (First Lemma)

The errors of the Schwarz waveform relaxation algorithm are bounded by the steady state solutions; we have for $j = 1, 2$ and all k

$$|e_j^{k+1}(x,t)| \leq \tilde{e}_j^{k+1}(x) \quad \forall x \in \Omega_j, \ t \in [0,\infty).$$

Proof.

The difference $d_j(x,t) := \tilde{e}_j^{k+1}(x) - e_j^{k+1}(x,t)$ satisfies

$$\begin{aligned}\partial_t d_j(x,t) &= \partial_t(\tilde{e}_j^{k+1}(x) - e_j^{k+1}(x,t)) = -\partial_t e_j^{k+1}(x,t)\\ &= -\partial_{xx} e_j^{k+1}(x,t) = \partial_{xx}(\tilde{e}_j^{k+1}(x) - e_j^{k+1}(x,t))\\ &= \partial_{xx} d_j(x,t),\end{aligned}$$

On the initial line $d_j(x,0) = 0$, and on the boundary $d_1(0,t) = 0$ and $d_1(\beta,t) = ||e_1^{k+1}(\beta,\cdot)||_\infty - e_1^{k+1}(\beta,t) \geq 0$. Maximum principle $\Longrightarrow d_1(x,t) \geq 0$ (also $d_2(x,t) \geq 0$). Analogously from the sum $\tilde{d}_j(x,t) := \tilde{e}_j^{k+1}(x) + e_j^{k+1}(x,t)$ we obtain $\tilde{d}_1(x,t) \geq 0$ and $\tilde{d}_2(x,t) \geq 0$. Therefore

$$\begin{aligned}d_j(x,t) &= \tilde{e}_j^{k+1}(x) - e_j^{k+1}(x,t) \geq 0,\\ \tilde{d}_j(x,t) &= \tilde{e}_j^{k+1}(x) + e_j^{k+1}(x,t) \geq 0,\end{aligned}$$

which implies that

$$-\tilde{e}_j^{k+1}(x) \leq e_j^{k+1}(x,t) \leq \tilde{e}_j^{k+1}(x),$$

and thus the bound in modulus.

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Lemma (Second Lemma)

The solutions of the steady upper bound satisfy

$$\begin{aligned}\tilde{e}_1^{k+1}(\alpha) &= \frac{\alpha}{\beta}||e_1^{k+1}(\beta,\cdot)||_\infty\\ \tilde{e}_2^{k+1}(\beta) &= \frac{L-\beta}{L-\alpha}||e_2^{k+1}(\alpha,\cdot)||_\infty.\end{aligned}$$

Proof.

The solutions of the steady problems are simply the affine functions

$$\tilde{e}_1^{k+1}(x) = \frac{x}{\beta}||e_1^{k+1}(\beta,\cdot)||_\infty, \quad \tilde{e}_2^{k+1}(x) = \frac{L-x}{L-\alpha}||e_2^{k+1}(\alpha,\cdot)||_\infty,$$

as one can see by inspection, and thus the result follows by evaluating at $x = \alpha$ and $x = \beta$. □

Theorem (Linear Convergence Estimate)

For any $0 < \alpha < \beta < L$, the parallel Schwarz waveform relaxation algorithm converges, and satisfies

$$\sup_{\substack{t\in[0,\infty)\\ x\in\Omega_j}} |u(x,t)-u_j^{2k}(x,t)| \leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k \sup_{t\in[0,\infty)} |u(\Gamma_j,t)-u_j^0(\Gamma_j,t)|$$

where $\Gamma_1 := \beta$ and $\Gamma_2 := \alpha$.

Proof. Using the first and second Lemma, we obtain

$$\sup_{t\in[0,\infty)} |e_1^{k+1}(\alpha,t)| \leq \tilde{e}_1^{k+1}(\alpha) = \frac{\alpha}{\beta}||e_1^{k+1}(\beta,\cdot)||_\infty,$$

$$\sup_{t\in[0,\infty)} |e_2^{k+1}(\beta,t)| \leq \tilde{e}_2^{k+1}(\beta) = \frac{L-\beta}{L-\alpha}||e_2^{k+1}(\alpha,\cdot)||_\infty.$$

Transmission conditions in parallel Schwarz WR:

$$e_1^{k+1}(\beta,t) = e_2^k(\beta,t), \quad e_2^{k+1}(\alpha,t) = e_1^k(\alpha,t)$$

Proof continued

$$||e_1^{k+1}(\alpha,\cdot)||_\infty \leq \frac{\alpha}{\beta}||e_2^k(\beta,\cdot)||_\infty \leq \frac{\alpha}{\beta}\frac{L-\beta}{L-\alpha}||e_1^{k-1}(\alpha,\cdot)||_\infty,$$

$$||e_2^{k+1}(\beta,\cdot)||_\infty \leq \frac{L-\beta}{L-\alpha}||e_1^k(\alpha,\cdot)||_\infty \leq \frac{\alpha}{\beta}\frac{L-\beta}{L-\alpha}||e_2^{k-1}(\beta,\cdot)||_\infty.$$

Now using again the transmission conditions $e_2^{k+2}(\alpha,t) = e_1^{k+1}(\alpha,t)$ and $e_1^{k+2}(\beta,t) = e_2^{k+1}(\beta,t)$ on the left, and $e_2^k(\alpha,t) = e_1^{k-1}(\alpha,t)$ and $e_1^k(\beta,t) = e_2^{k-1}(\beta,t)$ on the right, we obtain by induction

$$||e_2^{2k}(\alpha,\cdot)||_\infty \leq \left(\frac{\alpha}{\beta}\frac{L-\beta}{L-\alpha}\right)^k ||e_2^0(\alpha,\cdot)||_\infty,$$

$$||e_1^{2k}(\beta,\cdot)||_\infty \leq \left(\frac{\alpha}{\beta}\frac{L-\beta}{L-\alpha}\right)^k ||e_1^0(\beta,\cdot)||_\infty.$$

The maximum principle implies that the error inside the subdomains is smaller than on the interfaces,

$$\sup_{\substack{t\in[0,\infty)\\ x\in\Omega_j}} |e_j^{2k}(x,t)| \leq ||e_j^{2k}(\Gamma_j,\cdot)||_\infty, \quad \Gamma_1 := \beta, \Gamma_2 := \alpha.$$

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Superlinear Convergence

Lemma (Third Lemma)

The errors of the Schwarz WR algorithm are bounded by

$$|e_j^{k+1}(x,t)| \leq \bar{e}_j^{k+1}(x,t),$$

where the functions $\bar{e}_j^{k+1}(x,t)$ *are solutions of*

$$\begin{array}{rcll}
\partial_t \bar{e}_1^{k+1}(x,t) &=& \partial_{xx}\bar{e}_1^{k+1}(x,t) & \text{in } (-\infty,\beta)\times(0,T],\\
\bar{e}_1^{k+1}(x,0) &=& 0 & \text{in } (-\infty,\beta),\\
\bar{e}_1^{k+1}(\beta,t) &=& ||e_2^k(\beta,\cdot)||_t & \text{in } (0,T],\\
\partial_t \bar{e}_2^{k+1}(x,t) &=& \partial_{xx}\bar{e}_2^{k+1}(x,t), & \text{in } (\alpha,\infty)\times(0,T],\\
\bar{e}_2^{k+1}(x,0) &=& 0 & \text{in } (\alpha,\infty),\\
\bar{e}_2^{k+1}(\alpha,t) &=& ||e_1^k(\alpha,\cdot)||_t & \text{in } (0,T].
\end{array}$$

Proof.

The difference $d_j(x,t) := \bar{e}_j^{k+1}(x,t) - e_j^{k+1}(x,t)$ (and sum) satisfy again homogeneous heat equations with zero IC and non-negative BC. Hence $\bar{e}_j^{k+1}(x,t)$ bounds $|e_j^{k+1}(x,t)|$. $\square$

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Study of SWR on unbounded spatial domains

$$\begin{aligned}
\partial_t u_1^{k+1}(x,t) &= \partial_{xx} u_1^{k+1}(x,t) + f(x,t) && \text{in } (-\infty,\beta)\times(0,T],\\
u_1^{k+1}(x,0) &= u_0(x) && \text{in } (-\infty,\beta),\\
u_1^{k+1}(\beta,t) &= u_2^k(\beta,t) && \text{in } (0,T],
\end{aligned}$$

$$\begin{aligned}
\partial_t u_2^{k+1}(x,t) &= \partial_{xx} u_2^{k+1}(x,t) + f(x,t), && \text{in } (\alpha,\infty)\times(0,T],\\
u_2^{k+1}(x,0) &= u_0(x) && \text{in } \Omega_2,\\
u_2^{k+1}(\alpha,t) &= u_1^k(\alpha,t) && \text{in } (0,T],
\end{aligned}$$

Theorem (Superlinear Convergence Estimate)

For any $0<\alpha<\beta<L$, the parallel SWR algorithm converges, and satisfies

$$||u_j^{2k}(\Gamma_j,\cdot)-u(\Gamma_j,\cdot)||_T \leq erfc(\frac{(\beta-\alpha)k}{\sqrt{T}})||u_j^0(\Gamma_j,\cdot)-u(\Gamma_j,\cdot)||_T,$$

where $\Gamma_1 := \beta$ and $\Gamma_2 := \alpha$ and $erfc(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-s^2}ds$.

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Proof.

Using the Laplace transform

$$\hat{f}(s) := \int_0^\infty f(t)e^{-st}dt$$

the error equations become

$$\begin{array}{rcll} s\hat{e}_1^{k+1}(x,s) & = & \partial_{xx}\hat{e}_1^{k+1}(x,s) & \text{in } (-\infty,\beta), \\ \hat{e}_1^{k+1}(\beta,s) & = & \hat{e}_2^{k}(\beta,s) & \\ \\ s\hat{e}_2^{k+1}(x,s) & = & \partial_{xx}\hat{e}_2^{k+1}(x,s), & \text{in } (\alpha,\infty), \\ \hat{e}_2^{k+1}(\alpha,s) & = & \hat{e}_1^{k}(\alpha,s). & \end{array}$$

The solutions which go to zero when x goes to $\pm\infty$ are

$$\hat{e}_1^{k+1}(x,s) = \hat{e}_2^{k}(\beta,s)e^{\sqrt{s}(x-\beta)},\ \hat{e}_2^{k+1}(x,s) = \hat{e}_1^{k}(\alpha,s)e^{-\sqrt{s}(x-\alpha)},$$

because $\Re(s) > 0$.

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Proof continued

We thus obtain when evaluating on the interfaces

$$\begin{aligned}\hat{e}_1^{k+1}(\alpha,s) &= \hat{e}_2^{k}(\beta,s)e^{\sqrt{s}(\alpha-\beta)}\\ \hat{e}_2^{k+1}(\beta,s) &= \hat{e}_1^{k}(\alpha,s)e^{-\sqrt{s}(\beta-\alpha)}\end{aligned}$$

and introducing one into the other leads to

$$\begin{aligned}\hat{e}_1^{k+1}(\alpha,s) &= e^{-2\sqrt{s}(\beta-\alpha)}\hat{e}_1^{k-1}(\alpha,s)\\ \hat{e}_2^{k+1}(\beta,s) &= e^{-2\sqrt{s}(\beta-\alpha)}\hat{e}_2^{k-1}(\beta,s).\end{aligned}$$

We thus obtain by induction

$$\begin{aligned}\hat{e}_1^{2k+1}(\alpha,s) &= e^{-2k\sqrt{s}(\beta-\alpha)}\hat{e}_1^{1}(\alpha,s)\\ \hat{e}_2^{2k+1}(\beta,s) &= e^{-2k\sqrt{s}(\beta-\alpha)}\hat{e}_2^{1}(\beta,s).\end{aligned}$$

Now from the transmission conditions, we have $\hat{e}_2^{2k+2}(\alpha,s)=\hat{e}_1^{2k+1}(\alpha,s)$ and $\hat{e}_1^{2k+2}(\beta,s)=\hat{e}_2^{2k+1}(\beta,s)$ which we can replace on the left, and similarly $\hat{e}_2^{2}(\alpha,s)=\hat{e}_1^{1}(\alpha,s)$ and $\hat{e}_1^{2}(\beta,s)=\hat{e}_2^{1}(\beta,s)$ which we replace on the right, which then leads to the relation

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Proof continued

$$\hat{e}_2^{2k}(\alpha,s) = (\rho(s))^k \hat{e}_2^0(\alpha,s), \quad \hat{e}_1^{2k}(\beta,s) = (\rho(s))^k \hat{e}_1^0(\beta,s),$$

with the convergence factor $\rho(s) := e^{-2k\sqrt{s}(\beta-\alpha)}$. Now the inverse Laplace transform of $\rho(s)$ is given by

$$G(t) = \frac{k(\beta-\alpha)}{\sqrt{\pi t^3}} e^{-\frac{k^2(\beta-\alpha)^2}{t}},$$

which one can obtain by a direct computation, or with Maple

```
with(inttrans);
G:=invlaplace(exp(-2*C*sqrt(s)),s,t) assuming positive;
```

The Convolution Theorem of Laplace transforms then gives

$$e_1^{2k}(\beta,t) = \int_0^t e_1^0(\beta,t-\tau)G(\tau)d\tau,$$

and we can thus bound

$$|e_1^{2k}(\beta,t)| \leq \int_0^t |e_1^0(\beta,t-\tau)G(\tau)|d\tau \leq ||e_1^0(\beta,\cdot)||_T \int_0^T G(\tau)d\tau$$

$$= \mathrm{erfc}(\frac{(\beta-\alpha)k}{\sqrt{T}})||e_1^0(\beta,\cdot)||_T \text{ (similarly for } |e_1^{2k}(\alpha,t)|)$$

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Bounded Spatial Domains

Corollary

The same superlinear convergence result also holds for bounded domains and subdomains.

Proof.

Using the Third Lemma, the errors in the Schwarz waveform relaxation algorithm on bounded domains are bounded by the errors obtained by the same iteration on unbounded domains. It suffices therefore to start the iteration in the unbounded domain iteration with the errors from the bounded domain iteration $||e_j^0(\Gamma_j, \cdot)||_t$ to obtain the result. □

The superlinear convergence result also holds in the more general situation of semilinear parabolic partial differential equations [Gander:1998], and in higher spatial dimensions [G, Zhao:2002].

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Comparison with classical WR

Schwarz waveform relaxation converges faster than classical waveform relaxation: using Stirlings formula $k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$, we obtain for the classical waveform relaxation

$$e^{L_1 T}\frac{(L_2 T)^k}{k!} \sim e^{L_1 T}\frac{(L_2 Te)^k}{\sqrt{2\pi k}}e^{-k\log k},$$

while we have for Schwarz waveform relaxation with $\delta := \beta - \alpha$ denoting the overlap

$$\mathrm{erfc}\left(\frac{\delta k}{\sqrt{T}}\right) \sim \frac{\sqrt{T}}{\sqrt{\pi}\delta k}e^{-\frac{\delta^2}{T}k^2},$$

These asymptotic results can also be obtained directly with Maple,

```
asympt((C*T)^k/k!,k,2) assuming positive;
asympt(erfc(k*d/sqrt(T)),k,2) assuming positive;
```


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Schwarz WR for the Wave Equation

$$\begin{aligned}
\partial_{tt} u(x,t) &= c^2 \partial_{xx} u(x,t) + f(x,t) && \text{in } (0,L)\times(0,T],\\
u(x,0) &= u_0(x) && \text{in } (0,L),\\
\partial_t u(x,0) &= \tilde{u}_0(x) && \text{in } (0,L),\\
u(0,t) &= g_0(t) && \text{in } (0,T],\\
u(L,t) &= g_L(t) && \text{in } (0,T].
\end{aligned}$$

Parallel SWR with $\Omega_1 = (0,\beta)$ and $\Omega_1 = (\alpha, L)$, $\alpha < \beta$:

$$\begin{aligned}
\partial_{tt} u_1^{k+1}(x,t) &= c^2 \partial_{xx} u_1^{k+1}(x,t) + f(x,t) && \text{in } \Omega_1\times(0,T],\\
u_1^{k+1}(x,0) &= u_0(x) && \text{in } \Omega_1,\\
\partial_t u_1^{k+1}(x,0) &= \tilde{u}_0(x) && \text{in } \Omega_1,\\
u_1^{k+1}(0,t) &= g_0(t) && \text{in } (0,T],\\
u_1^{k+1}(\beta,t) &= u_2^{k}(\beta,t) && \text{in } (0,T],
\end{aligned}$$

$$\begin{aligned}
\partial_{tt} u_2^{k+1}(x,t) &= c^2 \partial_{xx} u_2^{k+1}(x,t) + f(x,t), && \text{in } \Omega_2\times(0,T],\\
u_2^{k+1}(x,0) &= u_0(x) && \text{in } \Omega_2,\\
\partial_t u_2^{k+1}(x,0) &= \tilde{u}_0(x) && \text{in } \Omega_2,\\
u_2^{k+1}(L,t) &= g_L(t) && \text{in } (0,T],\\
u_2^{k+1}(\alpha,t) &= u_1^{k}(\alpha,t) && \text{in } (0,T].
\end{aligned}$$

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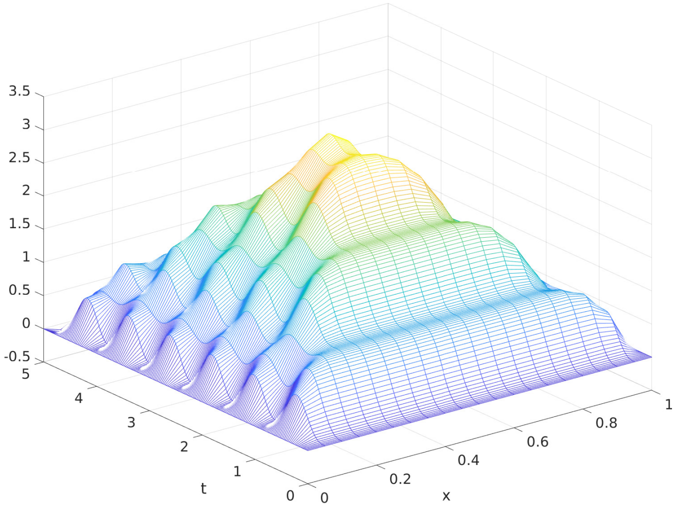
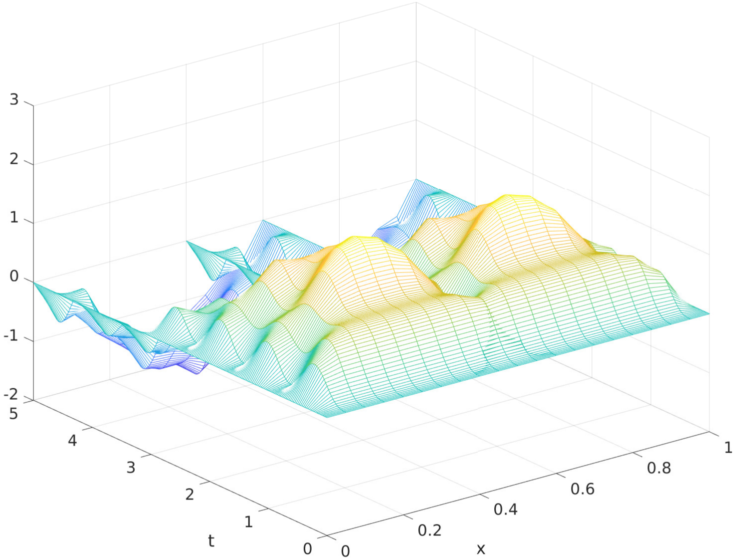
A Numerical Experiment

Source function $f(x,t) := x^4(1-x)^4 + 10\sin(8t)$ as for the heat equation. Solution for $T = 5$ and wave speed $c := 0.2$.



SWR iteration 1



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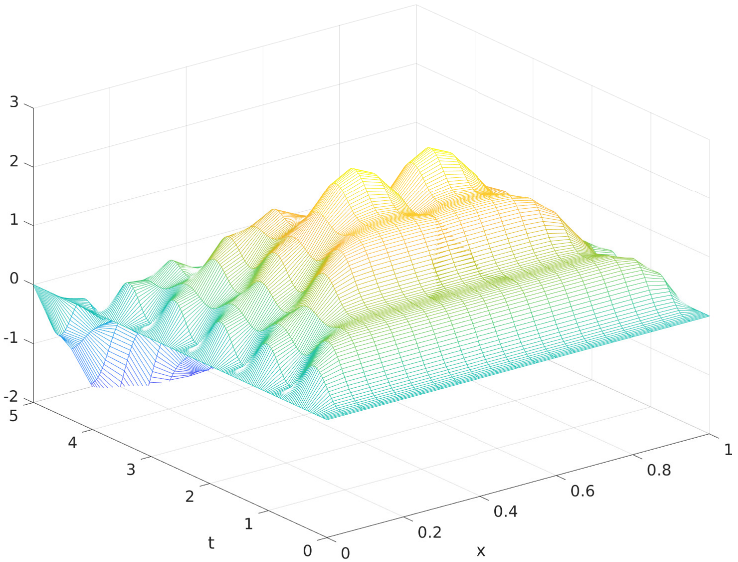
SWR iteration 2



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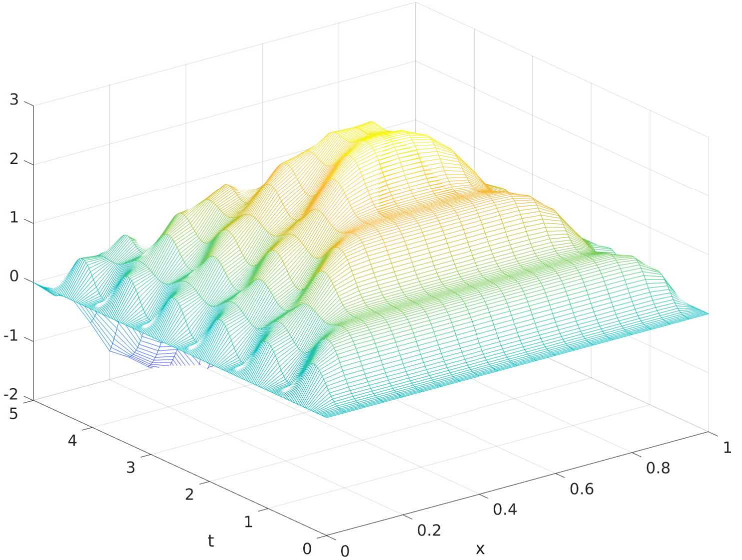
SWR iteration 3



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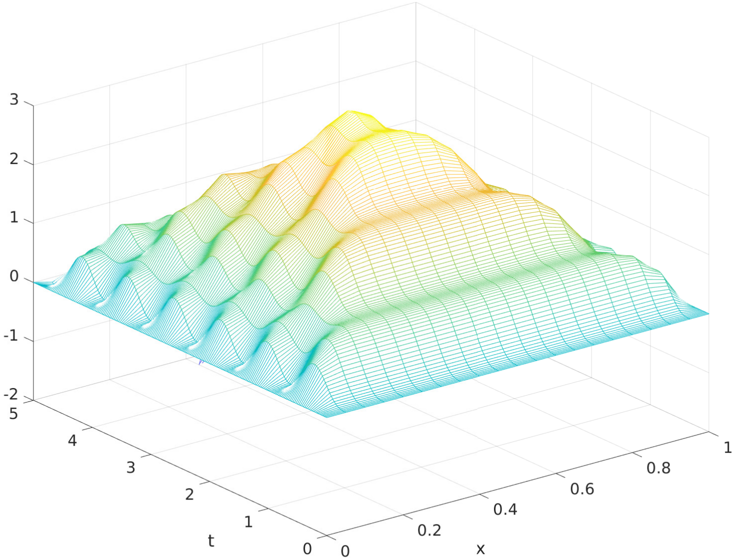
SWR iteration 4



PinT Summer School

Martin J. Gander

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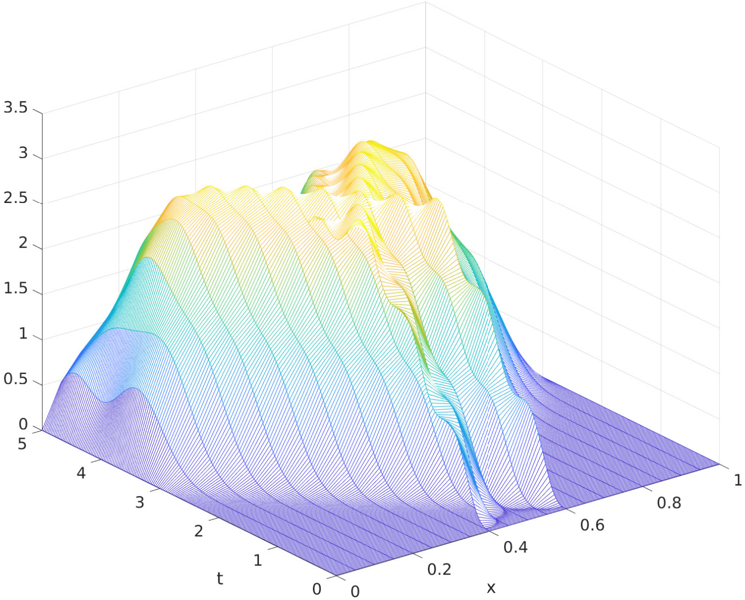
SWR error, iteration 1



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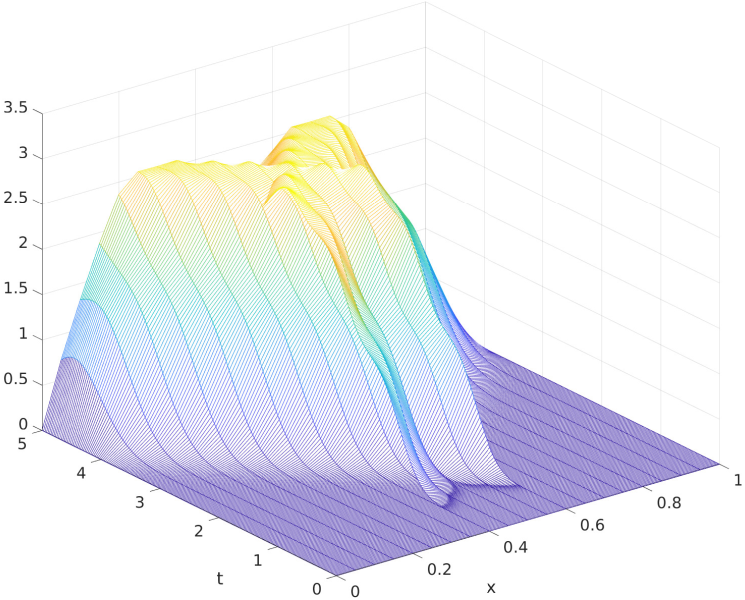
SWR error, iteration 2



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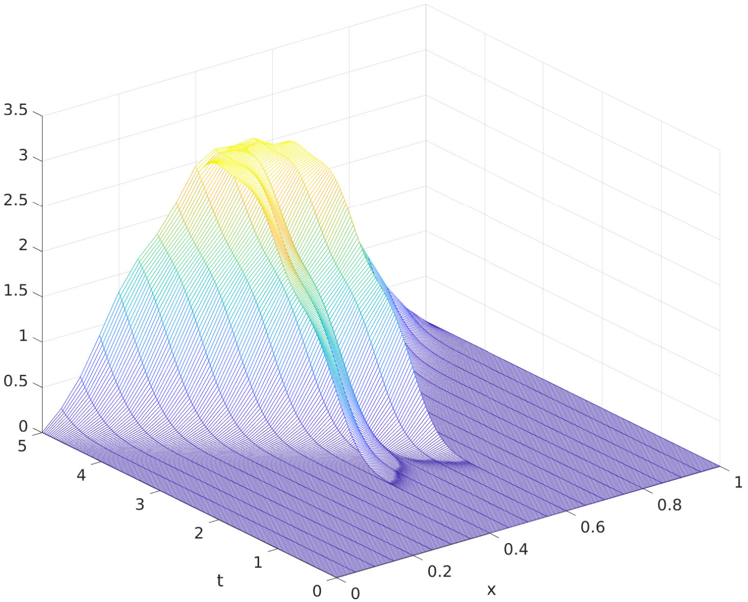
SWR error, iteration 3



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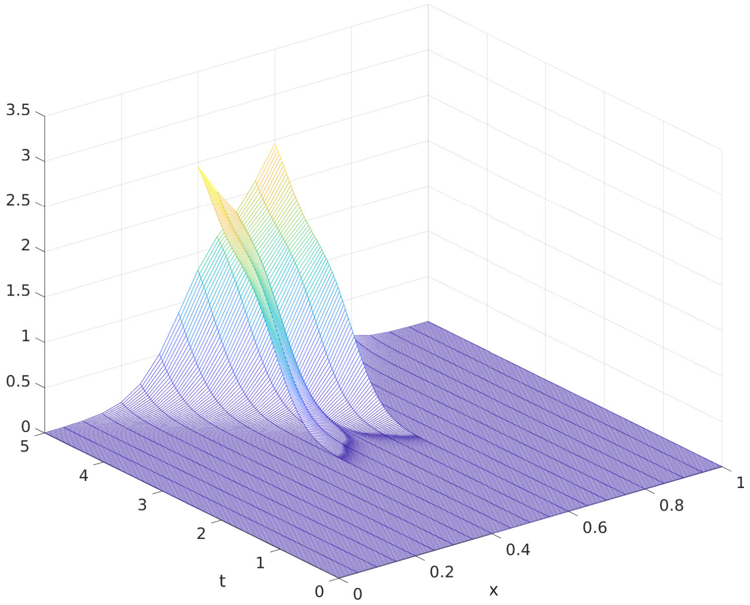
SWR error, iteration 4



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Convergence Analysis

Lemma

The solution of the wave equation with zero source term and initial conditions, $f(x,t) = 0$ and $u_0(x) = \tilde{u}_0(x) = 0$, and also $g_L(t) = 0$, is for $t \leq T := \frac{1}{c}L$ given by

$$u(x,t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{c}x, \\ g_0(t - \frac{1}{c}x) & \text{if } t > \frac{1}{c}x. \end{cases}$$

For a classical solution, $g_0(t)$ must also satisfy $g_0(0) = g_0'(0) = g_0''(0) = 0$.

If instead $g_0(t) = 0$ and $g_L(t) \neq 0$, then the solution is

$$u(x,t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{c}(L-x), \\ g_L(t + \frac{1}{c}(L-x)) & \text{if } t > \frac{1}{c}(L-x), \end{cases}$$

and again for a classical solution, we need $g_L(0) = g_L'(0) = g_L''(0) = 0$.

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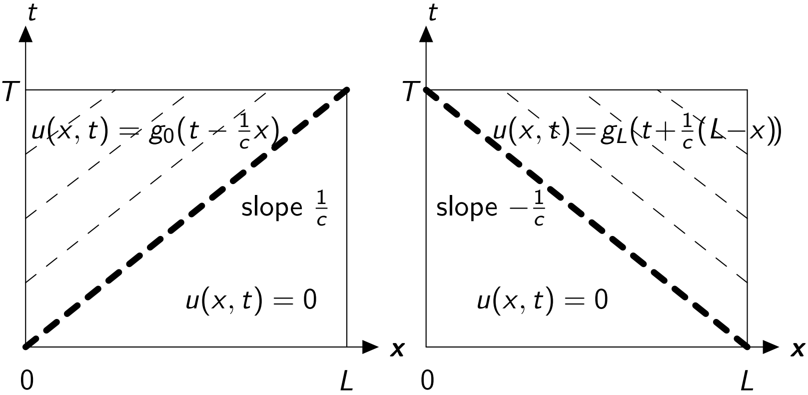
Proof.



For $g_0(t) \neq 0$, we obtain where the solution is non-zero

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \partial_{tt} g_0(t - \frac{1}{c}x) - c^2 \partial_{xx} g_0(t - \frac{1}{c}x) \\ &= g_0''(t - \frac{1}{c}x) - c^2 g_0''(t - \frac{1}{c}x)\frac{1}{c^2} = 0, \end{aligned}$$

and otherwise trivially $u_{tt} - c^2 u_{xx} = 0$.

Proof continued

Also accross the line $t = \frac{1}{c}x$ the equation holds because of the assumption that $g_0(0) = g_0'(0) = g_0''(0) = 0$.

Since on the boundary we also have

$$u(0,t) = g_0(t) \text{ for all } t, \text{ and } u(L,t) = 0 \text{ for } 0 \leq t \leq T = \frac{1}{c}L,$$

and on the initial line at $t = 0$

$$u(x,0) = \partial_t u(x,0) = 0,$$

the solution is the classical solution of this wave equation problem. Similarly also for the second case.

Note that for $t > T = \frac{1}{c}L$ the solution is not of this form any more, since there is a reflection from the zero boundary condition, so the Lemma only holds for $t \leq T$ as stated.

Convergence Result for the Wave Equation

Theorem (Convergence in a finite number of steps)

For the wave equation, the Schwarz waveform relaxation method converges in a finite number of steps: on the interfaces $x = \alpha, \beta$, *we have for* $0 \leq t \leq T$

$$u(\alpha, t) - u_1^k(\alpha, t) = u(\beta, t) - u_2^k(\beta, \cdot) = 0$$

as soon as $k > \frac{Tc}{\beta - \alpha}$.

This is a bit like Gaussian elimination, which also finishes the solution of linear systems in a finite number of steps!

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Proof.

The equations for the errors $e_j^{k+1}(x,t) := u(x,t) - u_j^{k+1}(x,t)$ satisfy

$$
\begin{array}{rcll}
\partial_{tt} e_1^{k+1}(x,t) & = & c^2 \partial_{xx} e_1^{k+1}(x,t) & \text{in } \Omega_1 \times (0,T], \\
e_1^{k+1}(x,0) & = & 0 & \text{in } \Omega_1, \\
\partial_t e_1^{k+1}(x,0) & = & 0 & \text{in } \Omega_1, \\
e_1^{k+1}(0,t) & = & 0 & \text{in } (0,T], \\
e_1^{k+1}(\beta,t) & = & e_2^k(\beta,t) & \text{in } (0,T],
\end{array}
$$

$$
\begin{array}{rcll}
\partial_{tt} e_2^{k+1}(x,t) & = & c^2 \partial_{xx} e_2^{k+1}(x,t), & \text{in } \Omega_2 \times (0,T], \\
e_2^{k+1}(x,0) & = & 0 & \text{in } \Omega_2, \\
\partial_t e_2^{k+1}(x,0) & = & 0 & \text{in } \Omega_2, \\
e_2^{k+1}(L,t) & = & 0 & \text{in } (0,T], \\
e_2^{k+1}(\alpha,t) & = & e_1^k(\alpha,t) & \text{in } (0,T].
\end{array}
$$

For $k = 0$, the interface errors $e_2^0(\beta,t)$ and $e_1^0(\alpha,t)$ are arbitrary, but the initial errors are zero and also on the boundaries at $x = 0, L$ the errors are zero.

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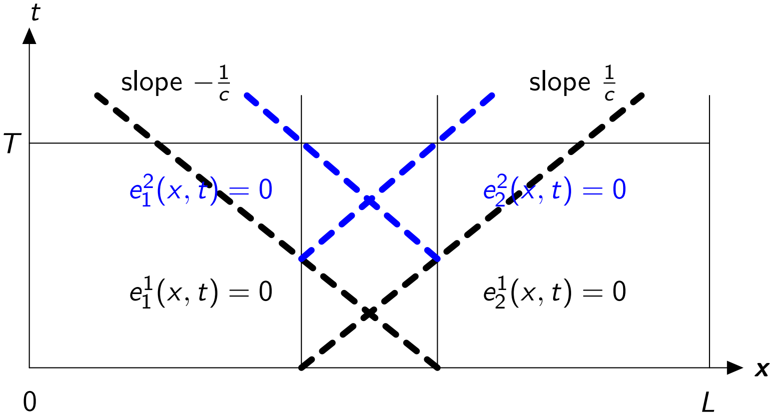
Proof continued

Using the Lemma

$$e_1^1(x,t) = 0 \text{ for } t \leq \frac{1}{c}(\beta - x), \quad e_2^1(x,t) = 0 \text{ for } t \leq \frac{1}{c}(x - \alpha)$$



Therefore on the interfaces for the next iteration, we have

$$e_1^1(\alpha,t) = 0 \text{ for } t \leq \frac{1}{c}(\beta - \alpha), \quad e_2^1(\beta,t) = 0 \text{ for } t \leq \frac{1}{c}(\beta - \alpha).$$

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Proof continued

This implies that below the two new blue characteristics, the errors $e_1^2(x,t)$ and $e_2^2(x,t)$ will be zero.

We thus obtain by induction on the interfaces that

$$e_1^k(\alpha,t) = 0 \text{ for } t \leq \frac{k}{c}(\beta-\alpha), \quad e_2^k(\beta,t) = 0 \text{ for } t \leq \frac{k}{c}(\beta-\alpha).$$

Hence if $k \geq \frac{Tc}{\beta-\alpha}$, we have $e_1^k(\alpha,t) = e_2^k(\beta,t) = 0$ for $t \leq T$ which concludes the proof.

Remarks:

- Result also holds for many subdomains and general hyperbolic equations
- For hyperbolic problems, there is no convergence result on long time windows
- This algorithms is related to tent pitching

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Optimized Schwarz Waveform Relaxation

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Better transmission conditions between subdomains; e.g. for the heat equation

$$\begin{aligned}
\partial_t u_1^{k+1}(x,t) &= \partial_{xx} u_1^{k+1}(x,t) + f(x,t) && \text{in } \Omega_1 \times (0,T],\\
u_1^{k+1}(x,0) &= u_0(x) && \text{in } \Omega_1,\\
u_1^{k+1}(0,t) &= g_0(t) && \text{in } (0,T],\\
(\partial_x + p)u_1^{k+1}(\beta,t) &= (\partial_x + p)u_2^{k}(\beta,t) && \text{in } (0,T],
\end{aligned}$$

$$\begin{aligned}
\partial_t u_2^{k+1}(x,t) &= \partial_{xx} u_2^{k+1}(x,t) + f(x,t), && \text{in } \Omega_2 \times (0,T],\\
u_2^{k+1}(x,0) &= u_0(x) && \text{in } \Omega_2,\\
u_2^{k+1}(L,t) &= g_L(t) && \text{in } (0,T],\\
(\partial_x - p)u_2^{k+1}(\alpha,t) &= (\partial_x - p)u_1^{k}(\alpha,t) && \text{in } (0,T].
\end{aligned}$$

In a more general setting, ∂_x represents the outer normal derivative ∂_n of the respective subdomain at the interface.

Convergence Analysis

Assume for simplicity as before that the domain $\Omega = \mathbb{R}$ and the subdomains are $\Omega_1 = (-\infty, \beta)$ and $\Omega_2 = (\alpha, \infty)$.

The error equations after a Laplace transform are

$$\begin{array}{rcll} s\hat{e}_1^{k+1}(x,s) & = & \partial_{xx}\hat{e}_1^{k+1}(x,s) & \text{in } (-\infty,\beta), \\ (\partial_x + p)\hat{e}_1^{k+1}(\beta,s) & = & (\partial_x + p)\hat{e}_2^{k}(\beta,s), & \end{array}$$

$$\begin{array}{rcll} s\hat{e}_2^{k+1}(x,s) & = & \partial_{xx}\hat{e}_2^{k+1}(x,s), & \text{in } (\alpha,\infty), \\ (\partial_x - p)\hat{e}_2^{k+1}(\alpha,s) & = & (\partial_x - p)\hat{e}_1^{k}(\alpha,s). & \end{array}$$

The solutions which go to zero when x goes to $\pm\infty$ are of the form

$$\hat{e}_1^{k+1}(x,s) = C_1^{k+1}(s)e^{\sqrt{s}(x-\beta)},\ \hat{e}_2^{k+1}(x,s) = C_2^{k+1}(s)e^{-\sqrt{s}(x-\alpha)}$$

because $\Re(s) > 0$.

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Convergence Factor

To determine C_j^{k+1}, $j = 1, 2$, use the transmission conditions: at $x = \beta$

$$(\partial_x + p)\hat{e}_1^{k+1}(\beta, s) = C_1^{k+1}(s)(\sqrt{s} + p) = (\partial_x + p)\hat{e}_2^{k}(\beta, s)$$
$$= C_2^{k}(s)(-\sqrt{s} + p)e^{-\sqrt{s}(\beta-\alpha)},$$

and similarly at $x = \alpha$

$$(\partial_x - p)\hat{e}_2^{k+1}(\alpha, s) = C_2^{k+1}(s)(-\sqrt{s} - p) = (\partial_x - p)\hat{e}_1^{k}(\alpha, s)$$
$$= C_1^{k}(s)(\sqrt{s} - p)e^{\sqrt{s}(\alpha-\beta)}.$$

Solving for the constant $C_2^k(s)$ at iteration k, we obtain

$$C_2^k(s) = \frac{\sqrt{s} - p}{-\sqrt{s} - p} e^{-\sqrt{s}(\beta-\alpha)} C_1^{k-1}(s).$$

Inserting this into the first relation yields

$$C_1^{k+1}(s) = \left(\frac{\sqrt{s} - p}{\sqrt{s} + p}\right)^2 e^{-2\sqrt{s}(\beta-\alpha)} C_1^{k-1}(s).$$

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Convergence Factor

Similarly also for the second constant,

$$C_2^{k+1}(s) = \left(\frac{\sqrt{s}-p}{\sqrt{s}+p}\right)^2 e^{-2\sqrt{s}(\beta-\alpha)} C_2^{k-1}(s).$$

The **convergence factor** of the optimized Schwarz waveform relaxation algorithm is therefore

$$\rho_R(s,p) = \left(\frac{\sqrt{s}-p}{\sqrt{s}+p}\right)^2 e^{-2\sqrt{s}(\beta-\alpha)},$$

and we obtain by induction for $j = 1, 2$

$$C_j^{2k}(s) = (\rho_R(s,p))^k\, C_j^0(s).$$

This implies for the Laplace transformed error functions

$$\hat{e}_1^{2k}(x,s) = (\rho_R(s,p))^k\, \hat{e}_1^0(x,s),\ \hat{e}_2^{2k}(x,s) = (\rho_R(s,p))^k\, \hat{e}_2^0(x,s).$$

Remark: $p := \sqrt{s}$ makes $\rho_R \equiv 0$, *optimal Schwarz waveform relaxation*, but $p := \sqrt{s}$ leads after a Laplace back-transform to non-local operators in time, the so-called Dirichlet to Neumann (DtN) operators.

Optimized Schwarz waveform relaxation

Use optimized local approximations of the DtN operators: take the Laplace transform formula

$$\hat{e}_j^k(x,s) = \int_0^\infty e_j^k(x,t)e^{-st}dt,$$

extend the errors $e_j^k(x,t)$ by zero continuously for $t<0$:

$$\hat{e}_j^k(x,s) = \int_{-\infty}^\infty e_j^k(x,t)e^{-st}dt = \int_{-\infty}^\infty e_j^k(x,t)e^{-\eta t}e^{-i\omega t}dt.$$

The Laplace transform can thus be interpreted as a Fourier transform in time of the weighted error functions $e_j^k(x,t)e^{-\eta t}$.

Using Parseval-Plancherel, we obtain for the L^2 norm the same if measured either in Laplace space or in time,

$$||\hat{e}_j^k(x,\eta+i\cdot)||_2 = ||e_j^k(x,\cdot)e^{-\eta\cdot}||_2.$$

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Weighted L^2 Convergence Estimate

$$
\begin{aligned}
||e_j^{2k}(x,\cdot)e^{-\eta\cdot}||_2 &= ||\hat{e}_j^{2k}(x,\eta+i\cdot)||_2 \\
&= ||(\rho_R(\eta+i\cdot,p))^{2k}\hat{e}_j^{0}(x,\eta+i\cdot)||_2 \\
&\leq \sup_{\omega\in\mathbb{R}}|\rho_R(\eta+i\omega,p)|^{2k}||\hat{e}_j^{0}(x,\eta+i\cdot)||_2 \\
&= \sup_{\omega\in\mathbb{R}}|\rho_R(\eta+i\omega,p)|^{2k}||e_j^{0}(x,\cdot)||_2.
\end{aligned}
$$

To make $\sup_{\omega\in\mathbb{R}}|\rho_R(\eta+i\omega,p)|$ small we set $\sqrt{s}:=x+iy$

$$
\sqrt{s}=\sqrt{\eta+i\omega}=\sqrt{\frac{\eta+\sqrt{\eta^2+\omega^2}}{2}}\pm i\sqrt{\frac{-\eta+\sqrt{\eta^2+\omega^2}}{2}}=:x\pm iy.
$$

We can then compute

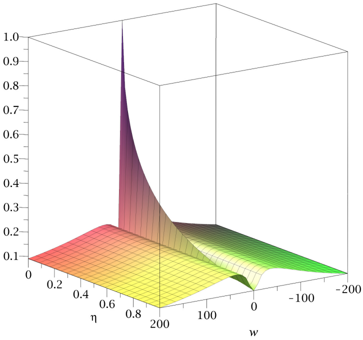
$$
\begin{aligned}
|\rho_R(s,p)|^2 &= \left|\frac{\sqrt{s}-p}{\sqrt{s}+p}\right|^2\left|e^{-\sqrt{s}(\beta-\alpha)}\right|^2 \\
&= \left|\frac{x+iy-p}{x+iy+p}\right|^2\left|e^{-(x+iy)(\beta-\alpha)}\right|^2 \\
&= \frac{(x-p)^2+y^2}{(x+p)^2+y^2}e^{-2x(\beta-\alpha)}.
\end{aligned}
$$

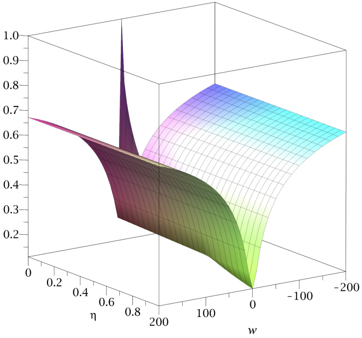
Convergence, even without overlap!

The convergence factor

$$|\rho_R(s,p)|^2 = \frac{(x-p)^2+y^2}{(x+p)^2+y^2}e^{-2x(\beta-\alpha)} < 1$$

for all $\omega \in \mathbb{R}$, if $\eta > 0$ and $p > 0$, since then $x > 0$ even if $\alpha = \beta$.





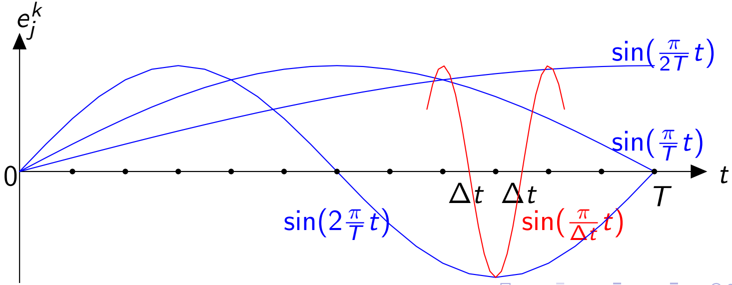
Convergence factor with overlap (left) and without (right).

Optimization of p

Need to solve the min-max problem

$$p = \underset{p>0}{\text{argmin}} \sup_{\omega\in\mathbb{R}} |\rho_R(\eta + i\omega, p)|^2.$$

- x and y are even functions of ω, consider only $\omega \geq 0$
- $\omega = 0$ is the constant mode in time, excluded by zero initial error.
- Heuristics for smallest and largest ω:



Numerically Relevant Min-Max Problem

We can minimize in the L^2 norm setting $\eta = 0$,

$$p = \underset{p>0}{\operatorname{argmin}} \sup_{\omega\in[\omega_{\min},\omega_{\max}]} |\rho_R(i\omega, p)|^2.$$

For $\eta = 0$, $x = \sqrt{\frac{\eta+\sqrt{\eta^2+\omega^2}}{2}}$ and $y = \sqrt{\frac{-\eta+\sqrt{\eta^2+\omega^2}}{2}}$ concide, so set

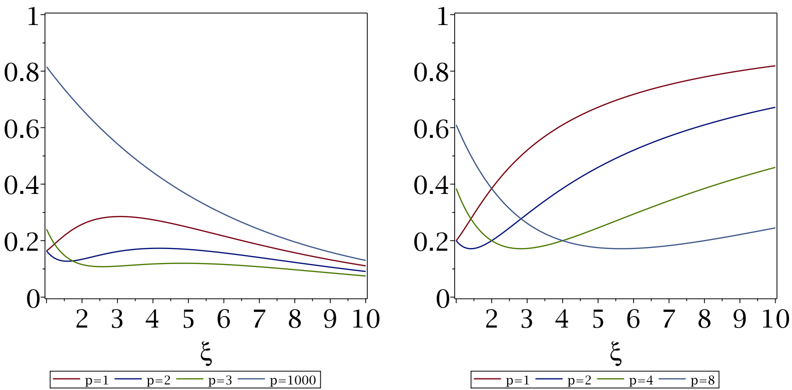
$$\xi := \sqrt{\frac{\omega}{2}} = x \equiv y,$$

and with overlap parameter $\delta := \beta - \alpha$, we get

$$R(\xi, p, \delta) := |\rho_R(i\omega, p)|^2 = \frac{(\xi - p)^2 + \xi^2}{(\xi + p)^2 + \xi^2} e^{-2\xi\delta},$$

and we need to optimize in the new parameter $\xi \in [\xi_{\min}, \xi_{\max}]$ with $\xi_{\min} := \sqrt{\frac{\omega_{\min}}{2}} = \sqrt{\frac{\pi}{4T}}$ and $\xi_{\max} := \sqrt{\frac{\omega_{\max}}{2}} = \sqrt{\frac{\pi}{2\Delta t}}$.

Examples of the Convergence Factor



Convergence factors, left with overlap, a good choice seems $p \approx 2$, and right without overlap a good choice is $p \approx 3$.

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Theorem (Optimized choice of p)

The solution of the min-max problem without overlap, $\delta = 0$, is given by

$$p^* = \sqrt{2\xi_{\min}\xi_{\max}},$$

and the associated convergence factor is bounded by

$$R(\xi_{\min}, p^*, 0) = \frac{\xi_{\max}+\xi_{\min}-p^*}{\xi_{\max}+\xi_{\min}+p^*}.$$

With overlap $\delta > 0$, the solution is for δ small given by

$$p^* \sim \left(\frac{\xi_{\min}^2}{\delta}\right)^{\frac{1}{3}},$$

and the associated convergence factor is bounded by

$$R(\xi_{\min}, p^*, \delta) = 1 - 4\xi_{\min}^{\frac{1}{3}}\delta^{\frac{1}{3}}.$$

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Proof.

Without overlap, $\delta = 0$, the solution problem is achieved by equioscillation,

$$R(\xi_{\min}, p^*, 0) = R(\xi_{\max}, p^*, 0),$$

as one can see from the derivative

$$\partial_p R(\xi, p, 0) = \frac{4\xi(p^2 - 2\xi^2)}{(p^2 + 2p\xi + 2\xi^2)^2},$$

which shows that when $p < \sqrt{2}\xi_{\min}$ increasing p decreases $R(\xi, p, 0)$ for all relevant $\xi \in [\xi_{\min}, \xi_{\max}]$. Similarly when $p > \sqrt{2}\xi_{\max}$, decreasing p decreases $R(\xi, p, 0)$ also for all relevant $\xi \in [\xi_{\min}, \xi_{\max}]$. Therefore the optimal p^* must lie in the interval $[\sqrt{2}\xi_{\min}, \sqrt{2}\xi_{\max}]$.

The derivative also shows that $R(\xi_{\min}, p, 0)$ increases when p starts increasing from $\sqrt{2}\xi_{\min}$, and $R(\xi_{\max}, p, 0)$ decreases. By continuity, the minimum is thus achieved by the equioscillation. Solving this equation gives directly the optimized choice p^* and resulting $R(\xi_{\max}, p^*, 0)$.

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Proof continued

With overlap $\delta > 0$, the solution is also by equioscillation,

$$R(\xi_{\min}, p^*, \delta) = R(\bar{\xi}, p^*, \delta),$$

where $\bar{\xi}$ is an interior maximum,

$$\bar{\xi} = \frac{\sqrt{p(1 + \sqrt{1 - \delta^2 p^2 - 2\delta p})}}{\sqrt{2\delta}}.$$

The equioscillation equation can however not be solved in closed form, only asymptotically, which leads to the results in the theorem.

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Maple is very useful here

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```
R:=((xi-p)^2+xi^2)/((xi+p)^2+xi^2)*exp(-2*xi*delta);
delta:=0;
factor(diff(R,p));
xi:=ximin;R1:=R;
xi:=ximax;R2:=R;
psols:=solve(R1=R2,p);
p:=psols[2];
simplify(R);
asympt(R,ximax,2);
```

where the last command gives

$$R(\xi_{\max}, p^*, 0) \sim 1 - 2\sqrt{2}\sqrt{\frac{\xi_{\min}}{\xi_{\max}}} = 1 - \frac{2^{\frac{7}{4}}\sqrt{\xi_{\min}}}{\pi^{\frac{1}{4}}}\Delta t^{\frac{1}{4}}.$$

More Maple for the Overlapping Case

```
R:=((xi-p)^2+xi^2)/((xi+p)^2+xi^2)*exp(-2*xi*delta);
xi:=ximin;R1:=R;               # keep first maximum
xi:='xi';
Rp:=simplify(diff(R,xi));
xi:=xib;
R2:=R;                         # keep second maximum
R2p:=Rp;                       # and derivative there
xi:='xi';
xisols:=solve(R2p,xib);        # find zero derivative
xib:=xisols[1];
p:=Cp*delta^(-1/3);            # educated guess for p*
se1:=series(R1,delta,1);
se2:=series(R2,delta,1) assuming positive;
Cpsols:=solve(op(2,se1)=op(2,se2),Cp);
Cp:=Cpsols[1];
p;                             # asymptotic optimized p*
se1;                           # and convergence factor
```


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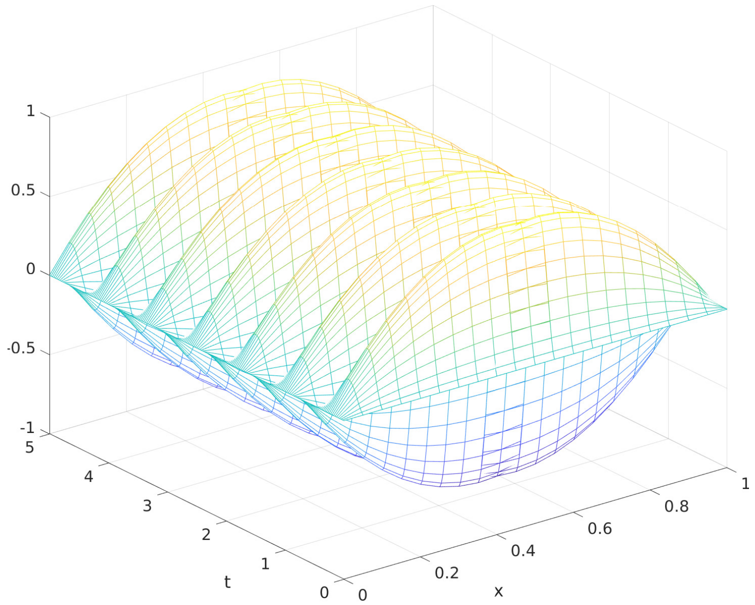
OSWR Iteration 1, $T = 5$



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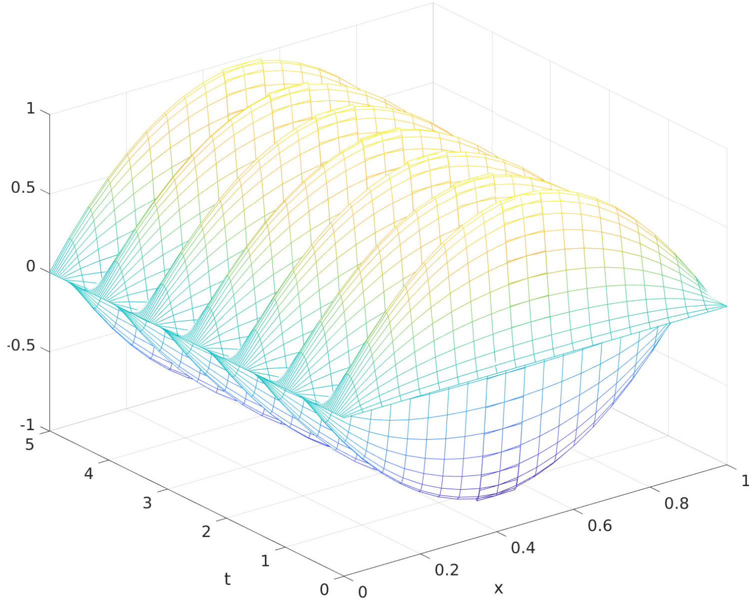
OSWR Iteration 2, $T = 5$



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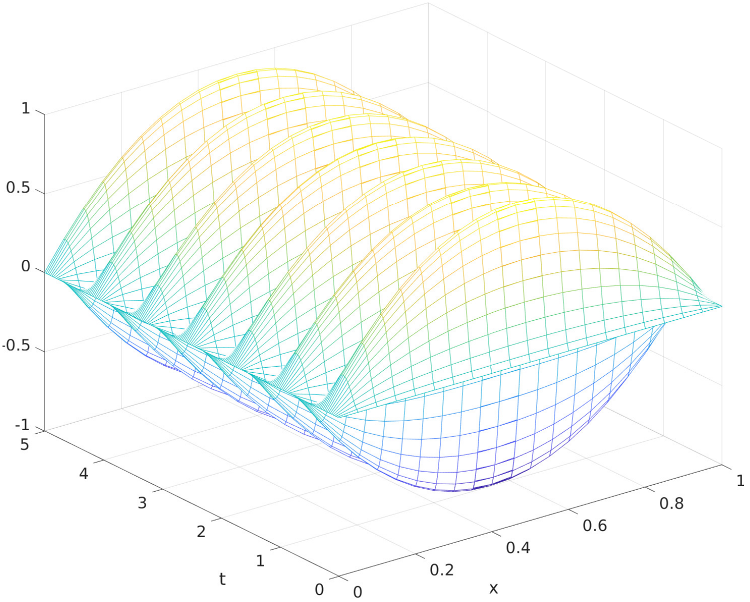
OSWR Iteration 3, $T = 5$



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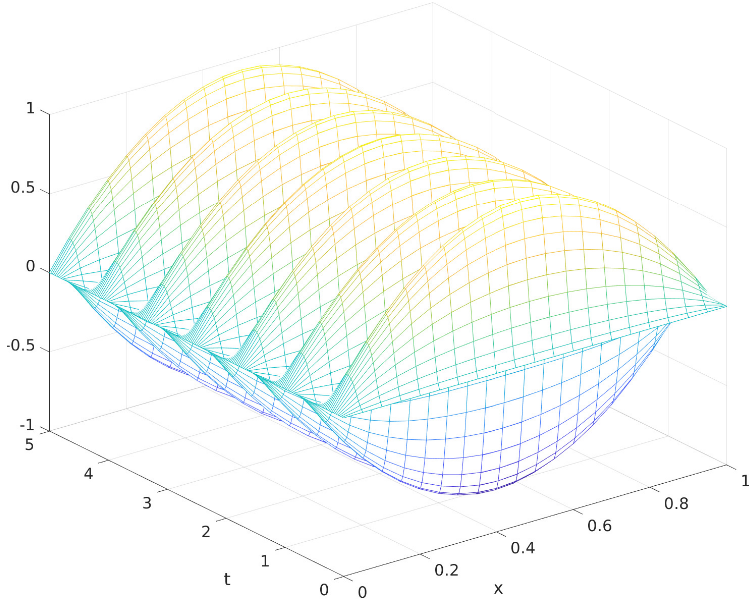
OSWR Iteration 4, $T = 5$



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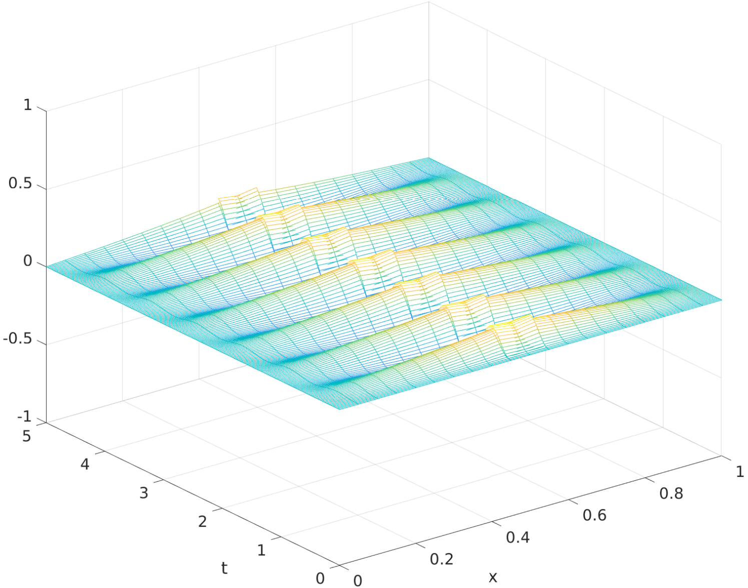
Error Iteration 1, $T = 5$



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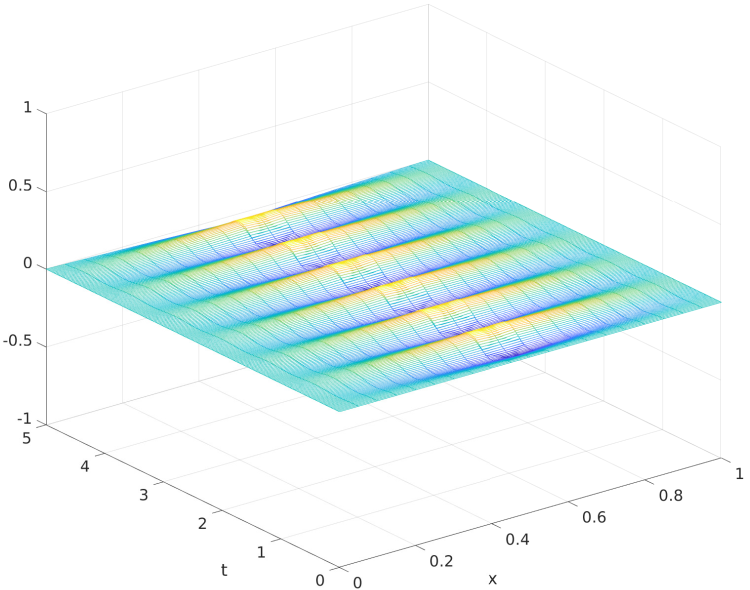
Error Iteration 2, $T = 5$



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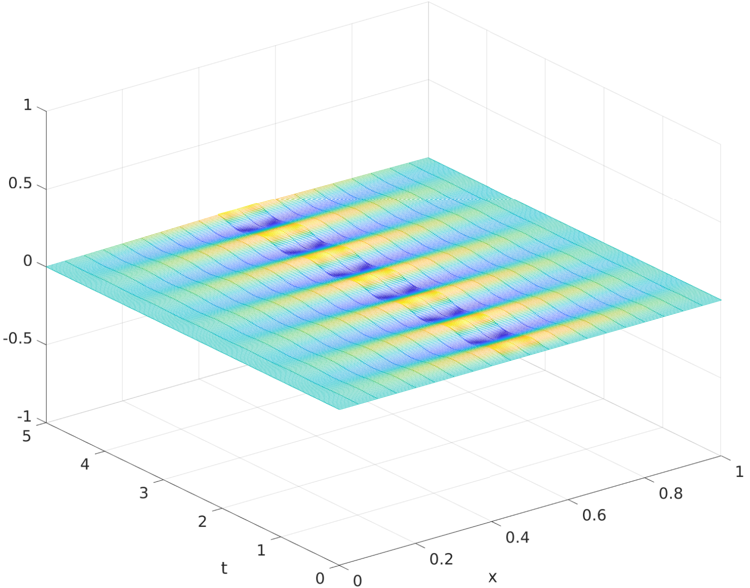
Error Iteration 3, $T = 5$



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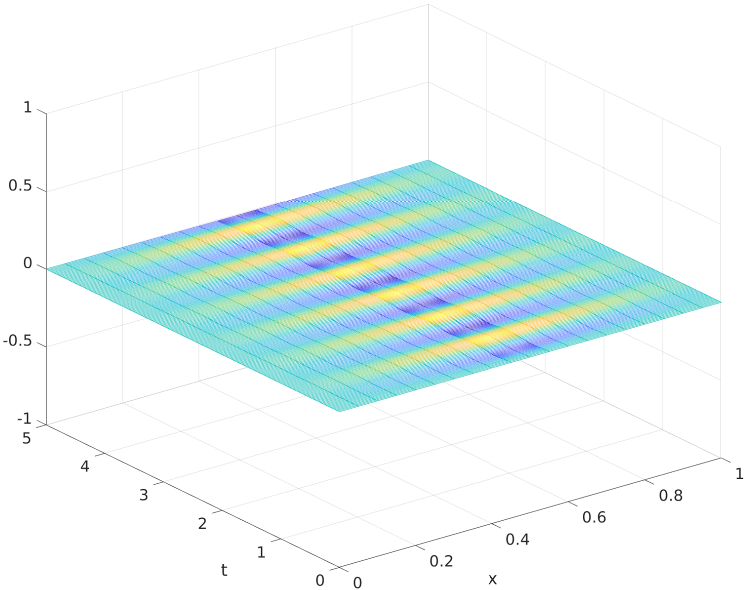
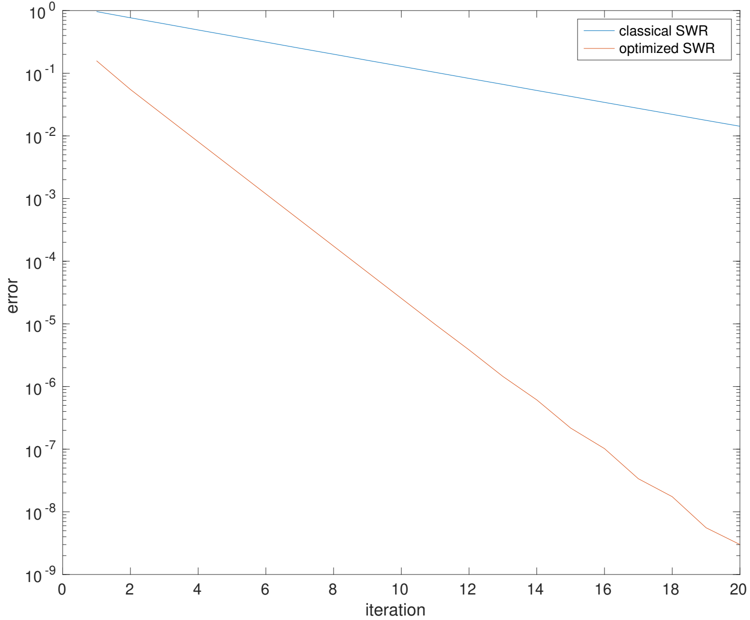


Comparison of classical and optimized SWR



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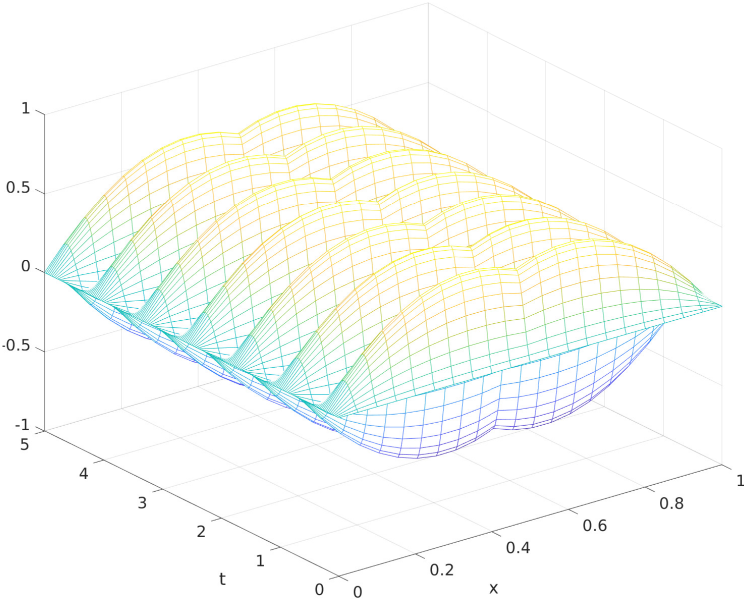
Non-overlapping OSWR, Iteration 1, $T = 5$



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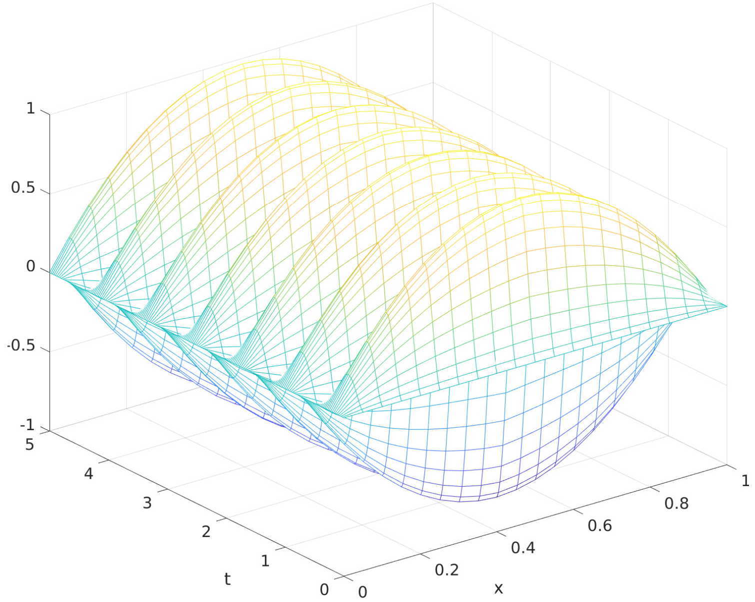
Non-overlapping OSWR, Iteration 2, $T = 5$



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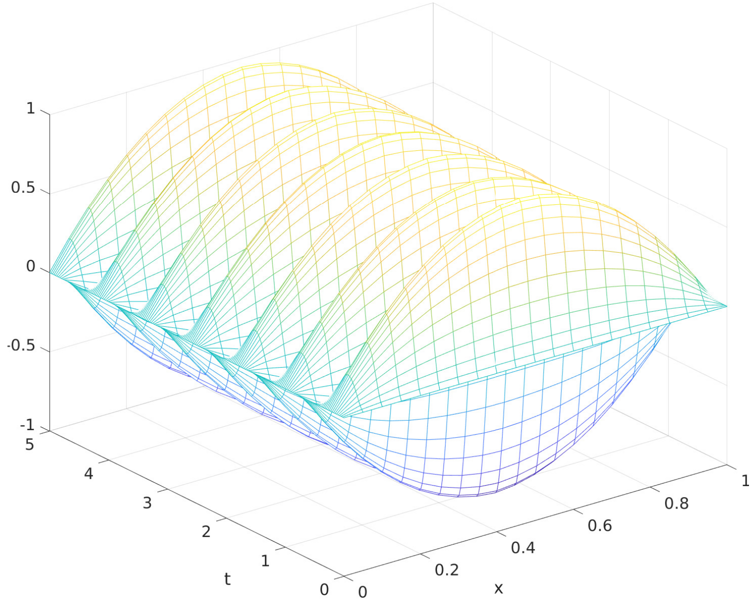

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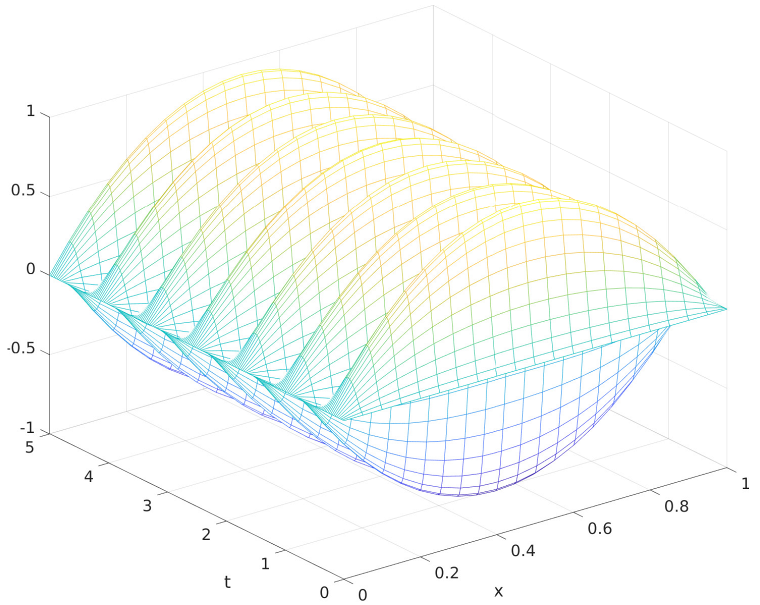
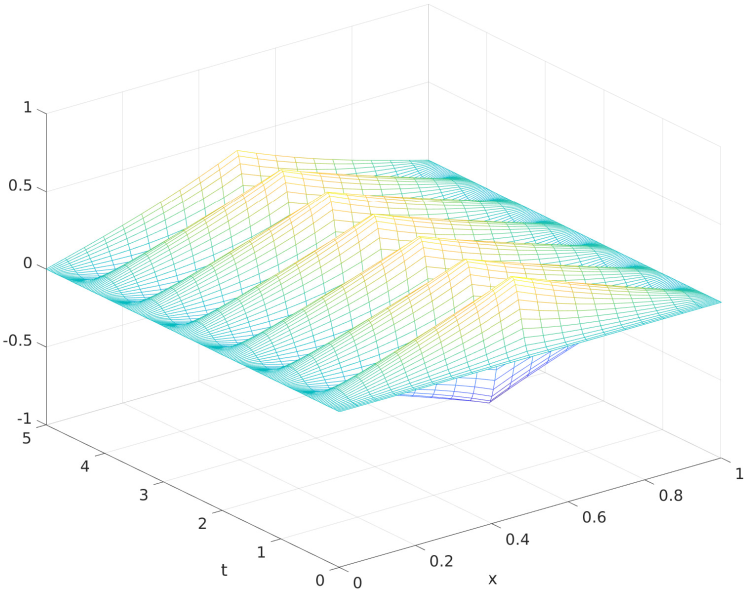


Non-overlapping OSWR error, Iteration 1, $T = 5$



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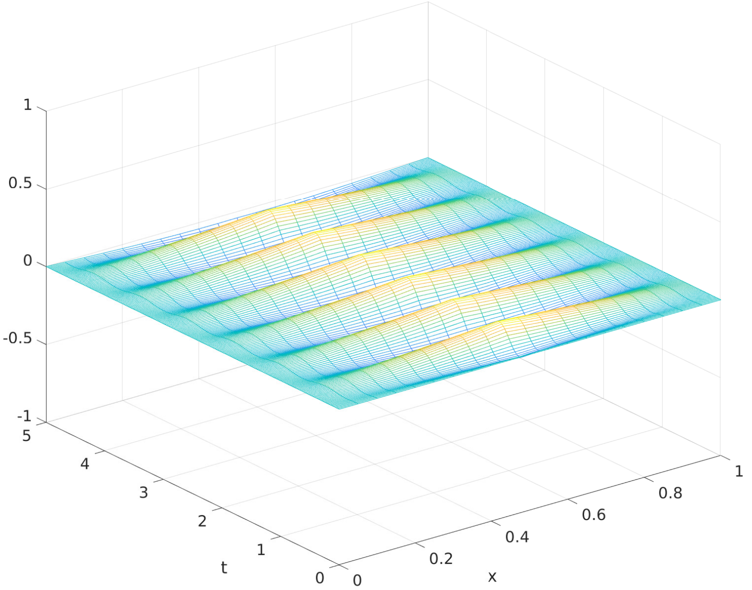
Non-overlapping OSWR error, Iteration 2, $T = 5$



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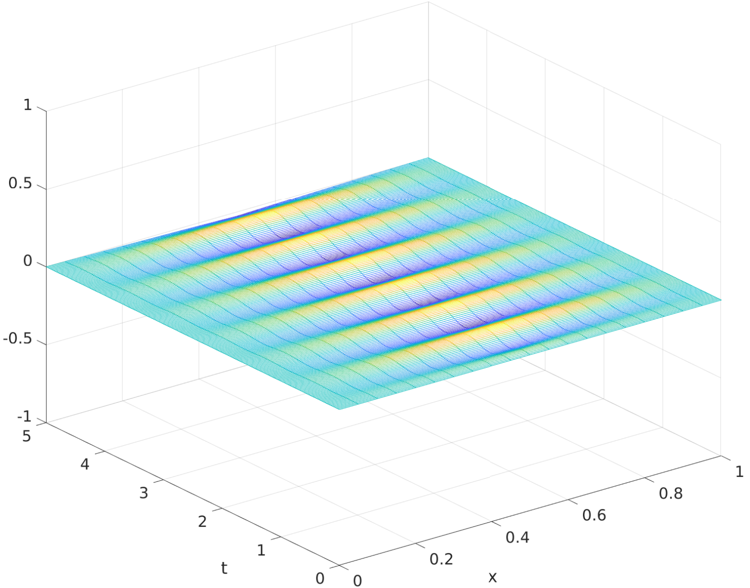

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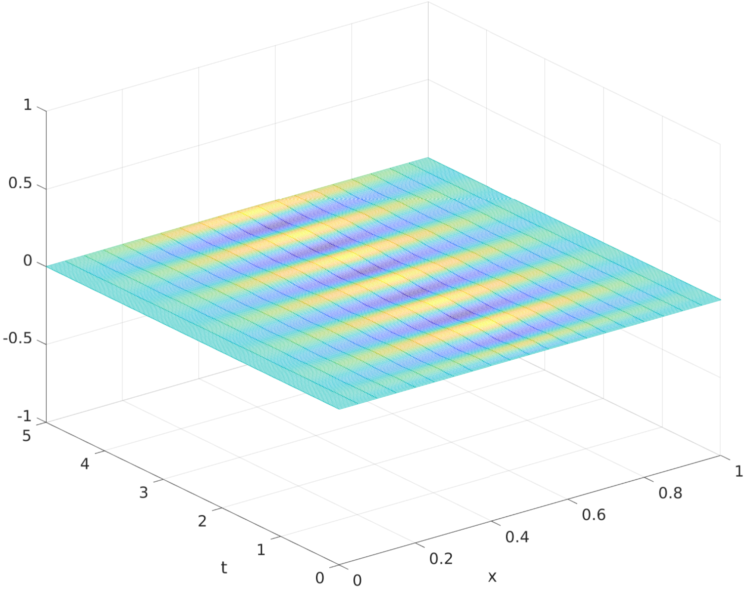

Convergence by Energy Estimates

Theorem

Without overlap, $\delta = \beta - \alpha = 0$, the Schwarz waveform relaxation algorithm on the domain $\Omega = (0, L)$ with subdomains $\Omega_1 = (0, \alpha)$ and $\Omega_2 = (\alpha, L)$ converges, i.e.

$$\lim_{k\to\infty} \sup_{t\in(0,T]} \int_{\Omega_j} (u(x,t) - u_j^k(x,t))^2 dx = 0.$$

Proof. We multiply the error equations

$$\begin{array}{rcll}
\partial_t e_1^k(x,t) & = & \partial_{xx} e_1^k(x,t) & \text{in } \Omega_1 \times (0,T], \\
e_1^k(x,0) & = & 0 & \text{in } \Omega_1, \\
e_1^k(0,t) & = & 0 & \text{in } (0,T], \\
(\partial_x + p) e_1^k(\alpha,t) & = & (\partial_x + p) e_2^{k-1}(\alpha,t) & \text{in } (0,T], \\
\partial_t e_2^k(x,t) & = & \partial_{xx} e_2^k(x,t), & \text{in } \Omega_2 \times (0,T], \\
e_2^k(x,0) & = & 0 & \text{in } \Omega_2, \\
e_2^k(L,t) & = & 0 & \text{in } (0,T], \\
(\partial_x - p) e_2^k(\alpha,t) & = & (\partial_x - p) e_1^{k-1}(\alpha,t) & \text{in } (0,T],
\end{array}$$

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Proof continued

by e_j^k and integrate over the domain Ω_j to obtain

$$\begin{aligned}\int_0^\alpha (\partial_t e_1^k(x,t))e_1^k(x,t)dx - \int_0^\alpha (\partial_{xx} e_1^k(x,t))e_1^k(x,t)dx &= 0,\\ \int_\alpha^L (\partial_t e_2^k(x,t))e_2^k(x,t)dx - \int_\alpha^L (\partial_{xx} e_2^k(x,t))e_2^k(x,t)dx &= 0.\end{aligned}$$

Now integration by parts in space, and using that integration in space and derivatives in time commute and that the errors on the outer boundaries vanish, we get

$$\frac{1}{2}\partial_t\int_0^\alpha (e_1^k(x,t))^2dx + \int_0^\alpha (\partial_x e_1^k(x,t))^2dx - (\partial_x e_1^k(\alpha,t))e_1^k(\alpha,t) = 0,$$

$$\frac{1}{2}\partial_t\int_\alpha^L (e_2^k(x,t))^2dx + \int_\alpha^L (\partial_x e_2^k(x,t))^2dx + (\partial_x e_2^k(\alpha,t))e_2^k(\alpha,t) = 0.$$

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Lions Trick

Now we use for the terms at the interface α the identity

$$ab = \frac{1}{4p}((a+pb)^2 - (a-pb)^2)$$

that holds for all $a, b \in \mathbb{R}$, and obtain

$$\begin{aligned}
\frac{1}{2}\partial_t \int_0^\alpha (e_1^k(x,t))^2 dx + \int_0^\alpha (\partial_x e_1^k(x,t))^2 dx& \\
+ \frac{1}{4p}(\partial_x e_1^k(\alpha,t) - p e_1^k(\alpha,t))^2& \\
= \frac{1}{4p}(\partial_x e_1^k(\alpha,t) + p e_1^k(\alpha,t))^2&, \\
\frac{1}{2}\partial_t \int_\alpha^L (e_2^k(x,t))^2 dx + \int_\alpha^L (\partial_x e_2^k(x,t))^2 dx& \\
+ \frac{1}{4p}(\partial_x e_2^k(\alpha,t) + p e_2^k(\alpha,t))^2& \\
= \frac{1}{4p}(\partial_x e_2^k(\alpha,t) - p e_2^k(\alpha,t))^2&.
\end{aligned}$$

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Proof continued

Using the transmission conditions, we can replace the right hand sides by terms from the previous iteration,

$$\frac{1}{4p}(\partial_x e_1^k(\alpha,t)+pe_1^k(\alpha,t))^2=\frac{1}{4p}(\partial_x e_2^{k-1}(\alpha,t)+pe_2^{k-1}(\alpha,t))^2,$$
$$\frac{1}{4p}(\partial_x e_2^k(\alpha,t)-pe_2^k(\alpha,t))^2=\frac{1}{4p}(\partial_x e_1^{k-1}(\alpha,t)-pe_1^{k-1}(\alpha,t))^2.$$

Now summing the two energy estimates with the rhs replaced yields

$$\begin{aligned}
&\frac{1}{2}\partial_t\int_0^\alpha (e_1^k(x,t))^2dx+\int_0^\alpha(\partial_x e_1^k(x,t))^2dx\\
&+\frac{1}{2}\partial_t\int_\alpha^L (e_2^k(x,t))^2dx+\int_\alpha^L(\partial_x e_2^k(x,t))^2dx\\
&+\frac{1}{4p}(\partial_x e_1^k(\alpha,t)-pe_1^k(\alpha,t))^2+\frac{1}{4p}(\partial_x e_2^k(\alpha,t)+pe_2^k(\alpha,t))^2\\
=&\frac{1}{4p}(\partial_x e_2^{k-1}(\alpha,t)+pe_2^{k-1}(\alpha,t))^2+\frac{1}{4p}(\partial_x e_1^{k-1}(\alpha,t)-pe_1^{k-1}(\alpha,t))^2
\end{aligned}$$

Proof continued

Now sum over the iteration index and get a telescopic sum, all interface terms cancel, except for the first and last one, and we get

$$
\begin{aligned}
&\sum_{k=1}^{K}\left(\frac{1}{2}\partial_t\int_0^{\alpha}(e_1^k(x,t))^2dx+\frac{1}{2}\partial_t\int_{\alpha}^{L}(e_2^k(x,t))^2dx\right)\\
&+\sum_{k=1}^{K}\left(\int_0^{\alpha}(\partial_x e_1^k(x,t))^2dx\int_{\alpha}^{L}(\partial_x e_2^k(x,t))^2dx\right)\\
&+\frac{1}{4p}(\partial_x e_1^K(\alpha,t)-pe_1^K(\alpha,t))^2+\frac{1}{4p}(\partial_x e_2^K(\alpha,t)+pe_2^K(\alpha,t))^2\\
=&\frac{1}{4p}(\partial_x e_2^0(\alpha,t)+pe_2^0(\alpha,t))^2+\frac{1}{4p}(\partial_x e_1^0(\alpha,t)-pe_1^0(\alpha,t))^2.
\end{aligned}
$$

Integrating now in time using the fact that the errors at time $t=0$ vanish, and neglecting the positive term at iteration index K on the left of the equal sign, we get the inequality

Proof continued

$$
\sum_{k=1}^{K}\frac{1}{2}\left(\int_0^{\alpha}(e_1^k(x,t))^2dx+\int_{\alpha}^{L}(e_2^k(x,t))^2dx\right)
$$

$$
+\sum_{k=1}^{K}\int_0^t\left(\int_0^{\alpha}(\partial_x e_1^k(x,\tau))^2dx+\int_{\alpha}^{L}(\partial_x e_2^k(x,\tau))^2dx\right)d\tau
$$

$$
\leq\frac{1}{4p}\int_0^t((\partial_x e_2^0(\alpha,\tau)+pe_2^0(\alpha,\tau))^2+(\partial_x e_1^0(\alpha,\tau)-pe_1^0(\alpha,\tau))^2)\,d\tau
$$

This inequality holds for all K, and since the right hand side is just a number independent of K, we can let K go to infinity, which shows that both terms in the sums on the left need to go to zero for the sums to remain bounded, since they are non-negative.

The result in the theorem then follows from the first of these sums.

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Optimized Schwarz WR for the Wave Equation

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$$
\begin{aligned}
\partial_{tt} u_1^{k+1}(x,t) &= c^2 \partial_{xx} u_1^{k+1}(x,t) + f(x,t) && \text{in } \Omega_1 \times (0,T],\\
u_1^{k+1}(x,0) &= u_0(x) && \text{in } \Omega_1,\\
\partial_t u_1^{k+1}(x,0) &= \tilde{u}_0(x) && \text{in } \Omega_1,\\
u_1^{k+1}(0,t) &= g_0(t) && \text{in } (0,T],\\
(\partial_x + p) u_1^{k+1}(\beta,t) &= (\partial_x + p) u_2^k(\beta,t) && \text{in } (0,T],
\end{aligned}
$$

$$
\begin{aligned}
\partial_{tt} u_2^{k+1}(x,t) &= c^2 \partial_{xx} u_2^{k+1}(x,t) + f(x,t), && \text{in } \Omega_2 \times (0,T],\\
u_2^{k+1}(x,0) &= u_0(x) && \text{in } \Omega_2,\\
\partial_t u_2^{k+1}(x,0) &= \tilde{u}_0(x) && \text{in } \Omega_2,\\
u_2^{k+1}(L,t) &= g_L(t) && \text{in } (0,T],\\
(\partial_x - p) u_2^{k+1}(\alpha,t) &= (\partial_x - p) u_1^k(\alpha,t) && \text{in } (0,T].
\end{aligned}
$$

Like for the heat equation, we consider the equations for the errors $e_j^k := u - u_j^k$, $j = 1,2$, which after a Laplace transform in time with with Laplace parameter s are

Error Equations in Laplace Space

$$\begin{aligned} s^2\hat{e}_1^{k+1}(x,s) &= c^2\partial_{xx}\hat{e}_1^{k+1}(x,s), \qquad \text{in } (-\infty,\beta),\\ (\partial_x+p)\hat{e}_1^{k+1}(\beta,s) &= (\partial_x+p)\hat{e}_2^{k}(\beta,s), \end{aligned}$$

$$\begin{aligned} s^2\hat{e}_2^{k+1}(x,s) &= c^2\partial_{xx}\hat{e}_2^{k+1}(x,s), \qquad \text{in } (\alpha,\infty),\\ (\partial_x-p)\hat{e}_2^{k+1}(\alpha,s) &= (\partial_x-p)\hat{e}_1^{k}(\alpha,s). \end{aligned}$$

Bounded solutions for $s := \eta + i\omega$ with $\eta \geq 0$ are of the form

$$\hat{e}_1^{k+1}(x,s) = C_1^{k+1}(s)e^{\frac{s}{c}(x-\beta)}, \quad \hat{e}_2^{k+1}(x,s) = C_2^{k+1}(s)e^{-\frac{s}{c}(x-\alpha)}.$$

To determine the constants C_j^{k+1}, $j = 1,2$, use the transmission conditions, at $x=\beta$ and $x=\alpha$

$$\begin{aligned} (\partial_x+p)\hat{e}_1^{k+1}(\beta,s) &= C_1^{k+1}(s)(\frac{s}{c}+p)\\ &= (\partial_x+p)\hat{e}_2^{k}(\beta,s) = C_2^{k}(s)(-\frac{s}{c}+p)e^{-\frac{s}{c}(\beta-\alpha)},\\ (\partial_x-p)\hat{e}_2^{k+1}(\alpha,s) &= C_2^{k+1}(s)(-\frac{s}{c}-p)\\ &= (\partial_x-p)\hat{e}_1^{k}(\alpha,s) = C_1^{k}(s)(\frac{s}{c}-p)e^{\frac{s}{c}(\alpha-\beta)}. \end{aligned}$$

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Solving for the constant $C_2^k(s)$ at iteration k, we obtain

$$C_2^k(s) = \frac{\frac{s}{c} - p}{-\frac{s}{c} - p} e^{-\frac{s}{c}(\beta-\alpha)} C_1^{k-1}(s).$$

Inserting this into the other relation yields

$$C_1^{k+1}(s) = \left(\frac{\frac{s}{c} - p}{\frac{s}{c} + p}\right)^2 e^{-2\frac{s}{c}(\beta-\alpha)} C_1^{k-1}(s),$$

and similarly also for the second constant,

$$C_2^{k+1}(s) = \left(\frac{\frac{s}{c} - p}{\frac{s}{c} + p}\right)^2 e^{-2\frac{s}{c}(\beta-\alpha)} C_2^{k-1}(s).$$

The *convergence factor* of the optimized Schwarz waveform relaxation algorithm for the wave equation is therefore

$$\rho_R(s,p) = \left(\frac{\frac{s}{c} - p}{\frac{s}{c} + p}\right)^2 e^{-2\frac{s}{c}(\beta-\alpha)},$$

very similar to the heat equation, only $\sqrt{s}$ is replaced by s/c, and we obtain by induction for $j = 1, 2$

$$C_j^{2k}(s) = (\rho_R(s,p))^k C_j^0(s).$$

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Local Optimal Schwarz Method

This implies for the Laplace transformed error functions

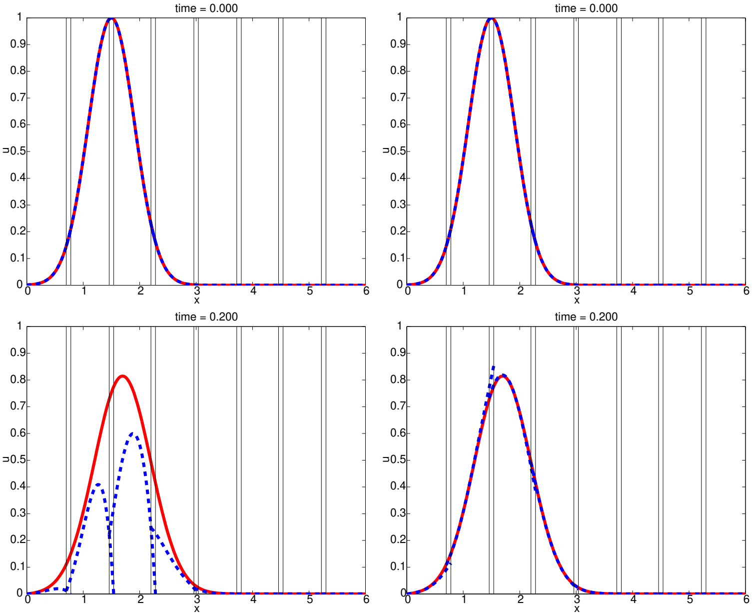
$$\begin{aligned}\hat{e}_1^{2k}(x,s) &= (\rho_R(s,p))^k \, \hat{e}_1^0(x,s),\\ \hat{e}_2^{2k}(x,s) &= (\rho_R(s,p))^k \, \hat{e}_2^0(x,s).\end{aligned}$$

We could choose $p := \frac{s}{c}$ for an optimal Schwarz waveform relaxation algorithm with a vanishing convergence factor, since $\partial_x \pm \frac{1}{c}\partial_t$ is local.

Many more Results:

- Wave equation in higher dimensions (G et al 2003, 2004)
- Advection reaction diffusion (G et al 2007, 2009)
- Laplace type problems (G 2006)
- Maxwell, Shallow Water, Circuits, ...
- Dirichlet-Neumann and Neumann-Neumann WR

Numerical Example, first iteration



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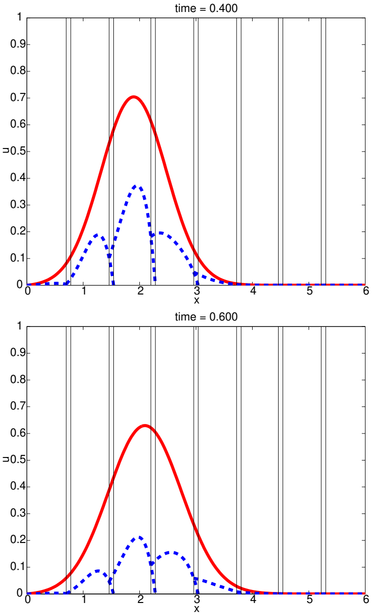
Convergence
OSWR for Waves
Parareal SWR

Numerical Example, first iteration



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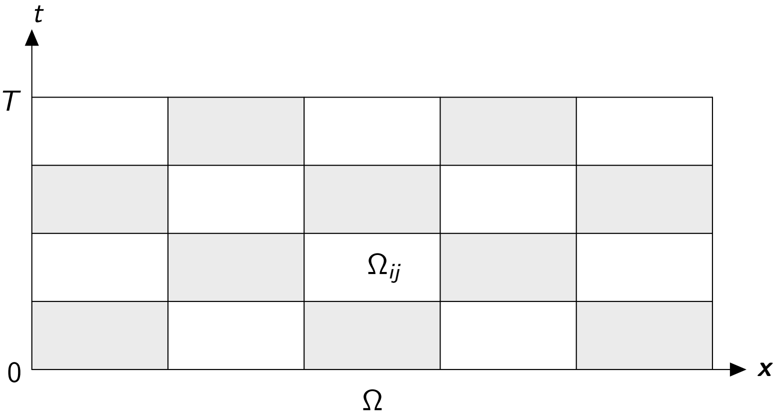
Parareal Schwarz Waveform Relaxation

Use a general space-time decomposition:



See Maday et al (2005) and G et al (2012, 2019)

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