Practical Session 1/3
First steps toward Parallel-in-Time

All of these problems are based on the slides provided during the morning sessions. Some additional source code (Python or Matlab) may be provided to help you with the implementation tasks.

Problem 1 *Study of the Lorenz equations*

1. *Compute the fixed points of the Lorenz system, as described in the slides. For each of those points, compute their stability properties by studying the eigenvalues of the Jacobi matrix of the Lorenz system at the fixed points.*
2. *Implement a generic solver called* `ForwardEuler` *to solve systems of ODEs using the Forward Euler method. It should be generic enough to solve any ODE of the form :*

$$\frac{du}{dt} = f(u,t), \quad u(t=0) = u_0, \quad t \in [0,T], \tag{1}$$

   where the right hand side f , the initial condition u_0 and the final time T are given as arguments for the solver.
3. *Solve the Lorenz system up to final time $T = 20$, using a Forward Euler numerical approximation. You may use the parameters*

$$\sigma = 10, \quad r = 28, \quad b = \frac{8}{3}, \tag{2}$$

   and the initial condition

$$x(0) = 20, \quad y(0) = 5, \quad z(0) = -5. \tag{3}$$

   Also, you can use $N = 20000$ time steps for the whole time interval. Plot each of the trajectories $x(t),\ y(t),\ z(t)$ separately, and then try to reproduce the "butterfly" curves by looking at the 3D trajectory.
4. *Using the same settings as in the previous question, add a small perturbation in the initial condition (for instance, $x(0) = 20 + \epsilon$ with $\epsilon = 1.0e^{-3}$). How small should the perturbation be in order to get the same trajectory for the whole simulation interval ?*
5. *Compute the local truncation error numerically for the same initial condition using a smaller and smaller time step. Plot the error as a function of the size of the time step using a log log plot. Can you see second order ?*
6. *Repeat the same experiment for the global truncation error, using as many time steps as needed for a short fixed time interval. Can you see first order ? Can you comment on the chosen time-step size initially used to solve the Lorentz system with Forward Euler ?*

Problem 2 *We consider the Dahlquist equation*

$$\frac{dy}{dt} = \lambda y, \;\; \lambda \in \mathbb{C}, \;\; t \in [0,T], \;\; y(0) = y_0. \tag{4}$$

We want to analyze the accuracy and the linear stability properties of the simplest classical time integration schemes, namely Forward Euler and Backward Euler. We note $i := \sqrt{-1}$.

1. *What is the exact solution of (4) ? Can you comment on the properties of the solution, depending on the value of λ ?*

2. *Show that the local truncation error is $O(\Delta t^2)$, and that the global truncation error is $O(\Delta t)$, for both time-integration schemes.*
3. *What is the region of absolute stability for both time-integration schemes ? Represent them in the complex plane, and comment on the numerical stability of Forward and Backward Euler for $\lambda \in \{-1, i, i-1\}$.*
4. *Using $T = 1$ and $N = 10$ time steps, represent the numerical solution for each $\lambda \in \{-1, i, i-1\}$. Comment on the error with the exact solution, and focus in particular on the phase and amplitude error of each numerical solution.*

Problem 3 *We consider the one-dimensional heat equation,*

$$\partial_t u = \partial_{xx} u + f(x,t), \quad x \in [0, L],\ t \in [0, T] \quad u(0,t) = u(L,t) = 0,\ u(x,0) = u_0(x) \tag{5}$$

and the one-dimensional transport (advection) equation,

$$\partial_t u = -a\partial_x u, \quad x \in [0, L],\ t \in [0, T] \quad u(0,t) = u(L,t)\ u(x,0) = u_0(x) \tag{6}$$

We want to analyze the link between those PDE and the Dahlquist equation, and implement a solver to obtain a numerical approximation of their solution.

1. *Use the method of lines to transform each PDE into a system of ODEs of the form*

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}(t) + \mathbf{f}(t), \tag{7}$$

   with A a matrix depending on the PDE, and $\mathbf{u}(t)$ a vector containing the discretization of the PDE solution on a given spatial mesh at a given time. You may use second order finite differences to approximate ∂_{xx} and first order upwind for ∂_x. What is the exact solution of (7) ? Is it the exact solution of the PDE ?
2. *What are the eigenvalues of the two A matrices for the heat and transport equation ? In which basis are both matrices diagonal ? Can you comment on the link with the Dahlquist equation, in particular looking at the form of the exact solution of (7) ?*
3. *How can you compute the numerical stability of Backward and Forward Euler, when used to numerically solve (7) ? Retrieve the stability condition for each problem and numerical scheme using the numerical stability condition computed for the Dahlquist equation. Compare with the CFL conditions for the transport equation as described in the slides.*
4. *Implement two generic time integration solvers for (7), using Forward and Backward Euler. The matrix A, the initial condition $u_0(x)$, and the source term $\mathbf{f}(t)$ should be given in arguments of the solver.*
5. *Solve the heat equation using $u_0(x) = 20$ as initial condition, $f(x,t) = 0$, $T = 1/2$ and $L = 1$. Represent the solution for $t \in [1/8, 1/4, 1/2]$.*
6. *Solve the transport equation using $u_0(x) = \sin(x)$, $L = 2\pi$, $a = 1$ and $T = 2\pi$. Compare with the exact solution of the advection equation. Can you find two configurations where the error is dominated either by the space discretization or by the time discretization ?*