A parallel-in-time IMEX multiderivative method

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Outline

- Introduction: Why MD-IMEX?

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- Predictor/corrector approach

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Low Mach

- Isentropic Euler equations

$$\rho_t + \nabla \cdot (\rho \boldsymbol{u}) = 0, \qquad (\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{\varepsilon^2} \nabla p(\rho) = 0$$

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$$\lambda_0 = \boldsymbol{u} \cdot \boldsymbol{n}, \qquad \lambda_\pm = \boldsymbol{u} \cdot \boldsymbol{n} \pm \frac{\sqrt{p'(\rho)}}{\varepsilon}$$

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- One slow wave ('convection'), two fast waves ('acoustics').

IMEX schemes

$$w_t + \nabla \cdot f_{\varepsilon}(w) = 0$$

*[Klein 1994], [Degond, Tang 2008], [Haack, Jin, Liu 2012], [Noelle et al. 2014], [Kaiser et al. 2017] ...

IMEX schemes

$$w_t + \nabla \cdot \left(\widetilde{f_\varepsilon}(w) + \widehat{f_\varepsilon}(w) \right) = 0$$

- $f_\varepsilon(w) = \widetilde{f_\varepsilon}(w) + \widehat{f_\varepsilon}(w)$.
- Ideal: $\widehat{f_\varepsilon}$ = convection, $\widetilde{f_\varepsilon}(w)$ = acoustics*.

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- Treat implicitly and explicitly :

$$\frac{w^{n+1} - w^n}{\Delta t} + \nabla \cdot \left(\widetilde{f_\varepsilon}(w^{n+1}) + \widehat{f_\varepsilon}(w^n) \right) = 0$$

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- This talk: Time integration $w'(t) = \Phi(w)$.

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IMEX schemes

$$\frac{w^{n+1} - w^n}{\Delta t} = \Phi_{\mathrm{I}}(w^{n+1}) + \Phi_{\mathrm{E}}(w^n)$$

- Higher order versions exist:
  - IMEX-RK
  - IMEX-BDF
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- Higher order versions exist:
  - IMEX-RK
  - IMEX-BDF
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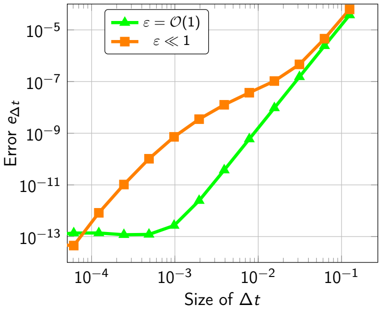
- Issues:
  - Storage requirements?
  - Really high order extensions?
  - Stability?
  - Parallel capabilities?
  - Asymptotic consistency/accuracy?
  - ...



Multiderivative approach

$$w'(t) = \Phi(w)$$

- 'Classical' methods: Use $w' \equiv \Phi(w)$ only.
- Multiderivatives: Use

$$\begin{aligned} w'' &\equiv \dot{\Phi}(w) := \Phi'(w)\Phi(w) \\ w''' &\equiv \ddot{\Phi}(w) := \ldots \end{aligned}$$

- Example: 3rd-order Euler:

$$w^{n+1} = w^n + \Delta t \Phi(w^n) + \frac{\Delta t^2}{2}\dot{\Phi}(w^n) + \frac{\Delta t^3}{6}\ddot{\Phi}(w^n).$$

Advertisement

$$w'(t) = \Phi(w)$$

- Advantages:
  - Storage efficient
  - Easy construction of high order, stable methods
  - Rather natural for PDEs
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Advertisement

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But: How to construct stable MD-IMEX schemes?

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Preliminaries: An integral approximation

$$w'(t) = \Phi(w), \qquad w''(t) = \dot{\Phi}(w)$$

- Two-derivative integral approximation:

$$\int_{t^n}^{t^{n+1}} \Phi(t)\mathrm{d}t = \underbrace{\frac{\Delta t}{2}\left(\Phi^n + \Phi^{n+1}\right) + \frac{\Delta t^2}{12}\left(\dot{\Phi}^n - \dot{\Phi}^{n+1}\right)}_{=:\mathcal{I}[\Phi^n;\Phi^{n+1}]} + \mathcal{O}(\Delta t^5)$$

- Relatively old ideas*.

* [Stroud and Stancu, 1965]

Preliminaries: Euler methods

$$w'(t) = \Phi(w) \equiv \Phi_{\mathrm{E}}(w) + \Phi_{\mathrm{I}}(w)$$

- *explicit* Euler (first order):

$$w^{n+1} = w^n + \Delta t \Phi(w^n)$$

- *implicit* Euler (first order):

$$w^{n+1} = w^n + \Delta t \Phi(w^{n+1})$$

- *IMEX* Euler (first order):

$$w^{n+1} = w^n + \Delta t \left(\Phi_{\mathrm{E}}(w^n) + \Phi_{\mathrm{I}}(w^{n+1}) \right)$$

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Preliminaries: Euler methods

$$w'(t) = \Phi(w) \equiv \Phi_E(w) + \Phi_I(w)$$

- *explicit* Euler (second order):

$$w^{n+1} = w^n + \Delta t \Phi(w^n) + \frac{\Delta t^2}{2} \dot{\Phi}(w^n)$$

- *implicit* Euler (second order):

$$w^{n+1} = w^n + \Delta t \Phi(w^{n+1}) - \frac{\Delta t^2}{2} \dot{\Phi}(w^{n+1})$$

- *IMEX* Euler (second order):

$$w^{n+1} = w^n + \Delta t \left(\Phi_E(w^n) + \Phi_I(w^{n+1}) \right) + \frac{\Delta t^2}{2} \left(\dot{\Phi}_E(w^n) - \dot{\Phi}_I(w^{n+1}) \right)$$

Overall method

1. Predict:

$$w^{[0]} := w^n + \Delta t \left(\Phi_{\mathsf{I}}(w^{[0]}) + \Phi_{\mathsf{E}}(w^n) \right) + \frac{\Delta t^2}{2} \left(\dot{\Phi}_{\mathsf{E}}(w^n) - \dot{\Phi}_{\mathsf{I}}(w^{[0]}) \right).$$

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2. Correct: $(0 \leq k \leq k_{\mathsf{max}} - 1)$

$$w^{[k+1]} := w^n + \Delta t \left(\Phi_{\mathsf{I}}^{[k+1]} - \Phi_{\mathsf{I}}^{[k]} \right) - \frac{\Delta t^2}{2} \left(\dot{\Phi}_{\mathsf{I}}^{[k+1]} - \dot{\Phi}_{\mathsf{I}}^{[k]} \right) + \mathcal{I}\left[\Phi^n; \Phi^{[k]} \right]$$

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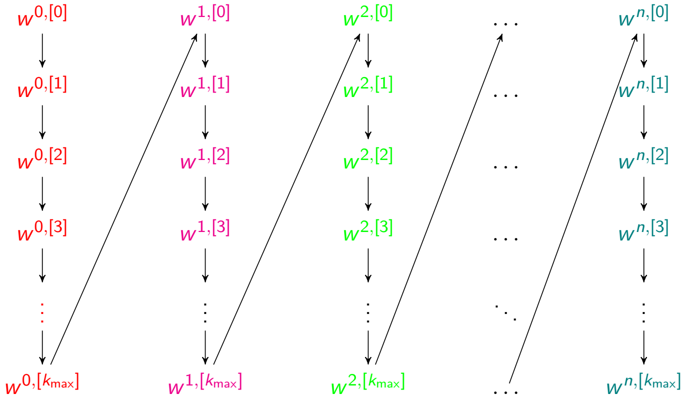
3. Update: Set $w^{n+1} := w^{[k_{\max}]}$.

How does it work?



- Different colors = different time intervals

Consistency analysis

- Classical consistency analysis*:

$$\frac{|w(t^{n+1}) - \tilde{w}^{n+1}|}{\Delta t} = \mathcal{O}\left(\Delta t^2 \cdot \Delta t^k\right) + \mathcal{O}\left(\Delta t^4\right).$$

- Second-order Euler predictor
- Correction
- Accuracy of $\mathcal{I}$.

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- Second-order Euler predictor
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- Accuracy of $\mathcal{I}$.
- Given $k_{\max} \geq 2$:

$$\frac{|w(t^{n+1}) - \tilde{w}^{n+1}|}{\Delta t} = \mathcal{O}\left(\Delta t^4\right).$$

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Consistency analysis (ctd.)

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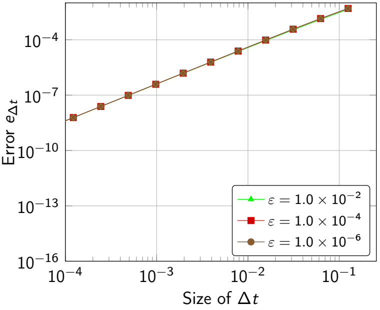

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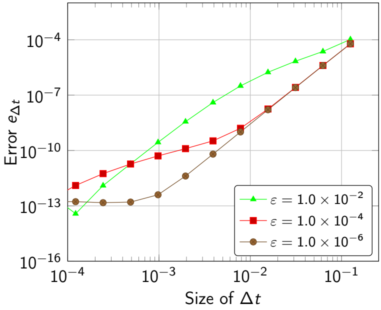

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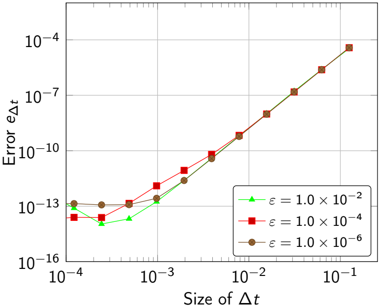

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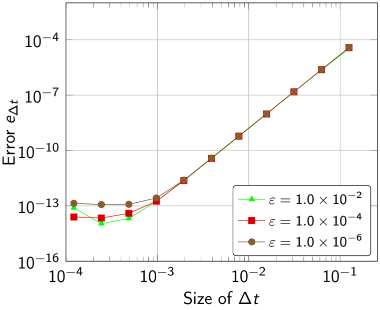

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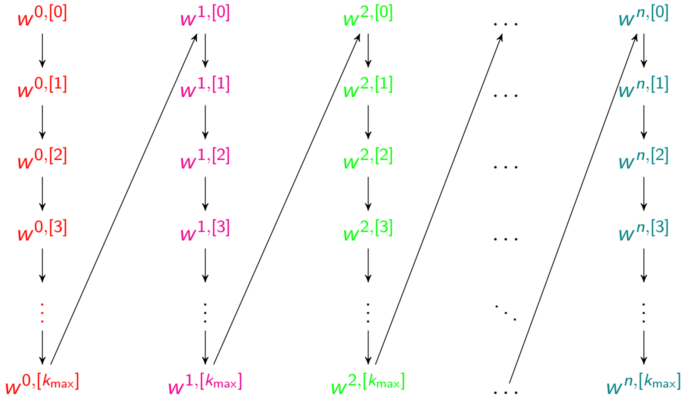
Method



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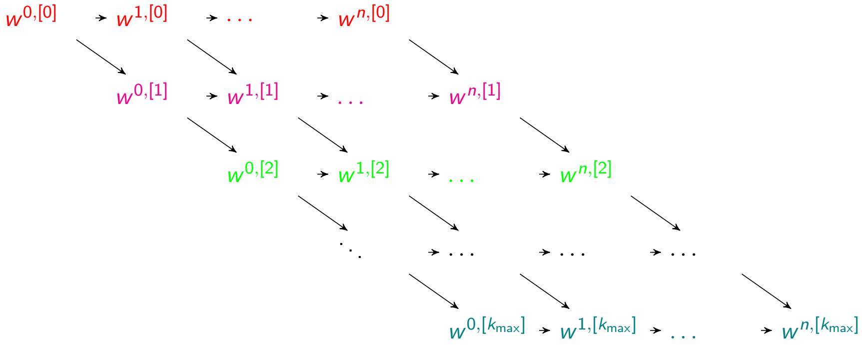
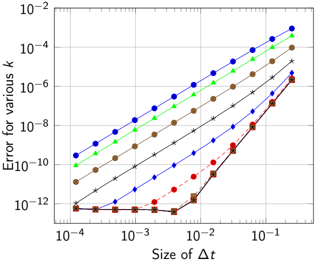
A time-parallel alternative

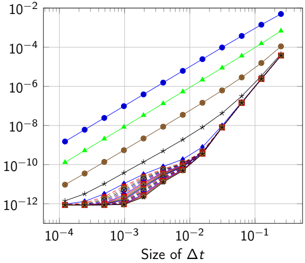


- Different colors = different processes.
- Not equivalent to original method!

Performance?

$$y'(t) = z, \qquad z'(t) = \frac{g(y,z)}{\varepsilon}$$





$\varepsilon = \mathbf{1}$ **(left) and** $\varepsilon = \mathbf{10^{-4}}$ **(right)**

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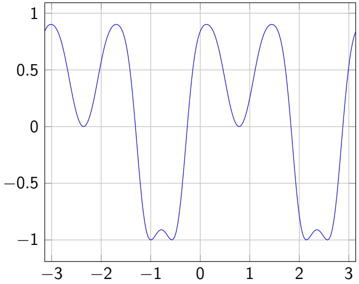
$$u(x,0) = \sin\left(\cos(4x) + \sin(2x)\right)$$



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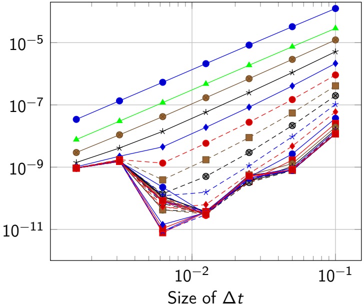

Spectral method, $N_k = 60$

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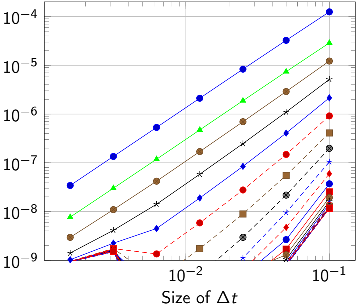

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Thank you!

Thank you for your attention!

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